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Detecting the Maximum of a Mean-Reverting Scalar Diffusion

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Detecting the Maximum of a Mean-Reverting Scalar Diffusion*

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Abstract

Let X be a mean reverting scalar process, X^* the corresponding running maximum, T_0 the first time X hits the level zero and ℓ a loss function, mainly increasing and convex. We consider the following optimal stopping problem:

$$\inf_{0\le\theta\le T_0} \mathbb{E}[\ell(X^*_{T_0} - X_\theta)],$$

over all stopping times θ with values in $[0, T_0]$. Under mild conditions, we prove that an optimal stopping time exists and is defined by:

$$\theta^* = \inf\{t \ge 0;\ X^*_t \ge \gamma(X_t)\},$$

where the boundary γ is explicitly characterized as the concatenation of the solutions of two equations. We investigate some examples such as the Ornstein-Uhlenbeck process, the CIR-Feller process, as well as the standard and drifted Brownian motions. Finally, we perform an empirical examination of the efficiency of this strategy on real financial data.

1 Introduction

Motivated by application in portfolio management, Graversen, Peskir and Shiryaev [5] considered the problem of detecting the maximum of a Brownian motion W on a fixed time period:. More precisely, denoting , [5] considers the optimal stopping problem:

$$\inf_{0\le\theta\le 1} \mathbb{E}[(W^*_1 - W_\theta)^p], \tag{1.1}$$

where $W^*_t := \max_{s\le t} W_s$ is the running maximum of W, $p > 0$ (and $p \neq 1$), and the infimum is taken over all stopping times θ taking values in $[0, 1]$. Using properties of the

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Brownian motion, [5] reduce the above problem into a one-dimensional infinite horizon optimal stopping problem, and prove that the optimal stopping rule is given by:

$$\theta^* := \inf\{t \le 1;\ W_t^* - W_t \ge b(t)\},$$

where the free boundary b is a explicit decreasing function.

A first extension of [5] was acheived by Pedersen [10], and later by Du Toit and Peskir [2], to the case of a Brownian motion with constant drift. A similar problem was solved by Shiryaev, Xu and Zhou [18] in the context of the exponential Brownian motion. See also Du Toit and Peskir [4] and Dai, Jin, Zhong and Zhou [1].

We also mention a connection with the problem of detection of the last moment τ when W reaches its maximum before the maturity $t = 1$:

$$\inf_{0 \le \theta \le 1} \mathbb{E}|\theta - \tau|.$$

This problem can indeed be related to the previous one by the observation of Urusov [19] that $\mathbb{E}(W_\tau - W_\theta)^2 = \mathbb{E}|\tau - \theta| + \frac{1}{2}$ for any stopping time θ, see Shiryaev in [17]. A similar problem formulated in the context of a drifted Brownian motion was solved by Du Toit and Peskir [3], although the latter identity is no longer valid.

In the present paper, we consider a scalar Markov diffusion X, which "mean-reverts" towards the origine starting from a positive initial data, and we consider the problem of optimal detection of the abosolute maximum up the the hitting time of the origin $T_0 := \int\{t \ge 0 : X_t = 0\}$:

$$\inf_{0 \le \theta \le T_0} \mathbb{E}[\ell(X_{T_0}^* - X_\theta)].$$

Here, the infimum is taken over all stopping times with values in $[0, T_0]$. We solve explicitly this problem as a free boundary problem. Our analysis has some similarities with that of Peskir [11], see also Obloj [9] and Hobson [6].

A major difficulty in the present context is that, in general, our solution exhibits a non-monotic free boundary made of two different parts, and driven by two different equations. Except for [3], the latter feature does not appear in the literature mentionned above, and has the following interpretation. Because of the mean-reversion, we expect that stopping is optimal whenever the drawdown $X^* - X$ is sufficiently large. On the other hand, if $X_t < X_t^*$ and X_t is small, we may expect that the martingale part of the process dominates the mean-reversion, so that it is not optimal to stop.

The paper is organized as follows. Section 2 presents the general framework and provides some necessary and sufficient conditions for the problem to be well defined. In Section 3, we derive the formulation as a free boundary problem, and we prove a verification result together with some preliminary properties. Sections 4 to 6 focus on the case of a quadratic loss function. In Section 4, we study a certain set Γ^+ which plays an essential role for the construction of the solution. The candidate boundary is exhibited in Section 5, and the corresponding candidate value function is shown to satisfy the assumptions of the verification result of section 3. Section 7 is dedicated to some examples. In Section 8, we provide sufficient conditions which guarantee that a similar solution is obtained for a general quadratic loss function. Finally, Appendix provides a possible trading strategy based on the results of this paper, and analyzes its performance using real financial data.

2 Problem formulation

Let W be a scalar Brownian motion on the complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and denote by $\mathbb{F} = \{\mathcal{F}_t, t \geq 0\}$ the corresponding augmented canonical filtration. Given two Lipschitz functions $\mu, \sigma : \mathbb{R} \longrightarrow \mathbb{R}$, we consider the scalar diffusion defined by the stochastic differential equation

$$dX_t \;=\; \mu(X_t)dt + \sigma(X_t)dW_t,\ t \geq 0,$$

together with some initial data $X_0 > 0$. We assume throughout that

$$\mu(x) \;\leq\; 0 \quad \text{for every} \quad x \geq 0, \tag{2.1}$$

meaning that the process X is reverted towards the origin, as well as $\sigma(x) > 0$ for every $x \geq 0$. For the purpose of this paper, the following stronger restrictions on the coefficients μ and σ are needed:

$$\text{the function} \quad \alpha := \frac{-2\mu}{\sigma^2} :\ (0,\infty) \longrightarrow \mathbb{R} \quad \text{is } C^2,\ \text{positive and concave.} \tag{2.2}$$

We introduce the so-called scale function S (see [7]):

$$S(x) := \int_0^x e^{\int_0^u \alpha(r)dr} du. \tag{2.3}$$

Remark 2.1 For later use, we observe that the restriction (2.2) has the following useful consequences:
(i) The function α is non-negative and non-decreasing. Consequently, $\int_0^u \alpha(r)dr < \infty$ and (2.3) is well-defined.
(ii) $(1/\alpha)'(x) \to 0$ as $x \to \infty$.
(iii) The function $2S' - \alpha S - 2$ is non-negative and increasing.

Notice that the mean reversion condition (2.1) is equivalent to the convexity of S, and implies that

$$\lim_{x\to\infty} S(x) \;=\; \infty. \tag{2.4}$$

We denote by

$$T_y \;:=\; \inf\{t > 0 :\ X_t = y\}$$

the first hitting time of the barrier y. We recall that, for a homogeneous scalar diffusion, we have

$$\mathbb{P}_x[T_y < T_0] = \frac{S(x)}{S(y)} \quad \text{for} \quad 0 \leq x < y, \tag{2.5}$$

Our main objective is to solve the optimization problem

$$V_0 \quad := \quad \inf_{\theta \in \mathcal{T}_0} \mathbb{E}\left[\ell\left(X^*_{T_0} - X_\theta\right)\right], \tag{2.6}$$

where $X^*_t := \max_{s \le t} X_s,\ t \ge 0$, is the running maximum process of X, $\ell : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is a non-decreasing, strcitly convex function, and $\mathcal{T}_0$ is the collection of all $\mathbb{F}-$stopping times θ with $\theta \le T_0$ a.s.

We shall approach this problem by the dynamic programming technique. We then introduce the dynamic version:

$$V(x,z) \quad := \quad \inf_{\theta \in \mathcal{T}_0} \mathbb{E}_{x,z}\left[\ell\left(Z_{T_0} - X_\theta\right)\right], \tag{2.7}$$

where $\mathbb{E}_{x,z}$ denotes the expectation operator conditional on $X_0 = x$ and $Z_0 = z$, and

$$Z_t := z \vee X^*_t, \qquad t \ge 0.$$

Clearly, the process (X, Z) takes values in the state space:

$$\mathbf{\Delta} := \{(x,z);\ 0 \le x \le z\}. \tag{2.8}$$

Defining the reward from stopping

$$g(x,z) := \mathbb{E}_{x,z}\left[\ell\left(Z_{T_0} - x\right)\right], \qquad (x,z) \in \mathbf{\Delta}, \tag{2.9}$$

we may re-write this problem in the standard form of an optimal stopping problem:

$$V(x,z) \quad := \quad \inf_{\theta \in \mathcal{T}_0} \mathbb{E}_{x,z}\left[g\left(X_\theta, Z_\theta\right)\right]. \tag{2.10}$$

Using (2.5), we immediately calculate that

$$\mathbb{P}_{x,z}[Z_{T_0} \le u] \;=\; \mathbb{P}_x[T_u \ge T_0]\mathbf{1}_{u \ge z} \quad = \quad \left(1 - \frac{S(x)}{S(u)}\right)\mathbf{1}_{u \ge z}, \tag{2.11}$$

so that

$$g(x,z) \quad = \quad \ell(z-x)\left(1 - \frac{S(x)}{S(z)}\right) + S(x)\int_z^\infty \ell(u-x)\frac{S'(u)}{S(u)^2}du \tag{2.12}$$

$$= \quad \ell(z-x) + S(x)\int_z^\infty \frac{\ell'(u-x)}{S(u)}du,\ 0 < x \le z, \tag{2.13}$$

where ℓ' is the generalized derivative of ℓ, and the latter expression is obtained by integration by parts together with the observation that

$$\int^\infty \ell(u)\frac{S'(u)}{S(u)^2}du < \infty \quad \text{iff} \quad \int^\infty \frac{\ell'(u)}{S(u)}du < \infty. \tag{2.14}$$

Indeed, since

$$\int_z^A \ell(u)\frac{S'(u)}{S(u)^2}du = \frac{\ell(z)}{S(z)} - \frac{\ell(A)}{S(A)} + \int_z^A \frac{\ell'(u)}{S(u)}du, \tag{2.15}$$

we clearly have $\int^\infty \ell(u)\frac{S'(u)}{S(u)^2}du = \infty$ implies $\int^\infty \frac{\ell'(u)}{S(u)}du = \infty$. Conversely, suppose that $\int^\infty \ell(u)\frac{S'(u)}{S(u)^2}du < \infty$. Then, since $S(\infty) = \infty$, it follows from (2.15) that $\int_z^\infty \ell(u)\frac{S'(u)}{S(u)^2}du \geq \ell(z)S(z)^{-1}$, and therefore $\ell(z)S(z)^{-1} \leq \int_1^\infty \ell(u)\frac{S'(u)}{S(u)^2}du$ for $z \geq 1$ by (2.4). Combined with (2.15), this shows that (2.14) holds true.

We now provide necessary and sufficient conditions on the loss function ℓ which ensure that V is finite on $\mathbb{R}_+$. Recall that $V(0,z) = g(0,z) = \ell(z)$ is always finite.

Proposition 2.1 *Assume that* $\alpha \geq 0$ *and*

$$\sup_{u \geq z} \frac{\ell(u)}{\ell(u-x)} < \infty \quad \textit{for every} \quad (x,z) \in \mathbf{\Delta}. \tag{2.16}$$

Then, the following statements are equivalent:
(i) $V(x,z) < \infty$ *for every* $0 \leq x \leq z$,
(i') $V(x_0,z_0) < \infty$ *for some* $0 < x_0 \leq z_0$,
(ii) $g(x,z) < \infty$ *for every* $0 \leq x \leq z$,
(ii') $g(x_0,z_0) < \infty$ *for some* $0 < x_0 \leq z_0$,
(iii) *either one of the equivalent conditions of* (2.14) *holds true.*

Proof. For $\theta \in \mathcal{T}_0$, set $J(\theta,x,z) := \mathbb{E}_{x,z}\ell(Z_{T_0} - X_\theta)$. The implications (ii) $\iff$ (ii') $\iff$ (iii) follow immediately from the definition of g in (2.12) together with Condition (2.16). Also the implications $(i) \implies (i')$ and $(ii) \implies (i)$ are immediate as $V \leq g$.

We conclude the proof by showing that $(i') \implies (iii)$. Let (i') hold true and assume to the contrary that $\int^\infty \frac{\ell'(u)}{S(u)}du = \infty$. For abritrary $0 < x \leq z$ and $\theta \in \mathcal{T}_0$, we have:

$$\mathbb{E}[\ell(Z_{T_0} - X_\theta)|X_\theta] = g(X_\theta, Z_\theta) = \begin{cases} +\infty \text{ if } X_\theta > 0, \\ \ell(Z_\theta) \text{ if } X_\theta = 0. \end{cases}$$

Let $A := \{\theta \neq T_0\}$. Then,

- either $\mathbb{P}(A) > 0$, and:

$$\begin{aligned} J(\theta,x,z) &= \mathbb{E}_{x,z}\ell(Z_{T_0} - X_\theta) = \mathbb{E}_{x,z}\mathbb{E}[\ell(Z_{T_0} - X_\theta)|X_\theta] \\ &\geq \mathbb{E}_{x,z}\mathbf{1}_A\mathbb{E}[\ell(Z_{T_0} - X_\theta)|X_\theta] = +\infty, \end{aligned}$$

- or $\mathbb{P}(A) = 0$, i.e. $\theta = T_0$ a.s. and $J(\theta,x,z) = J(T_0,x,z) = \ell(z) + S(x)\int_z^\infty \frac{\ell'(u)}{S(u)}du = +\infty$. By arbitrariness of $0 < x \leq z$ and $\theta \in \mathcal{T}_0$, this shows that $V = +\infty$ everywhere. □

Notice that if (2.14) holds, then (2.12) is also valid for $x = 0$.

Remark 2.2 Without assuming (2.16), we see from the previous proof that (2.14) is still a sufficient condition for (i) or (ii) to hold true. But in general, it is not a necessary condition. Indeed consider for example a process with scale function $S(x) = e^{x^2}$, and the loss function $\ell(x) = \int_0^x e^{u^2}du$. Then $\int_z^\infty \frac{\ell'(u)}{S(u)}du = +\infty$ while for $x > 0$, $\int_z^\infty \frac{\ell'(u-x)}{S(u)}du = \frac{e^{x^2+2xz}}{2x}$, so that (i) and (ii) are satisfied.

Remark 2.3 Condition (2.16) is satisfied by power and exponential loss functions $\ell(x) = x^p$ for some $p \geq 1$, or $e^{\eta x}$ for some $\eta > 0$. Without Condition (2.16), one cannot hope to prove that (i') $\Longrightarrow$ (i) or (ii') $\Longrightarrow$ (ii). Consider for instance the process with scale function $S(x) = e^{x^2}$ and, for $\varepsilon > 0$, the loss function $\ell(x) = \int_0^x e^{(u+\varepsilon)^2} du$. Then if $x \leq \varepsilon$, $\int_z^\infty \frac{\ell'(u-x)}{S(u)} du = \infty$, while if $x > \varepsilon$, $\int_z^\infty \frac{\ell'(u-x)}{S(u)} du = \frac{e^{(x-\varepsilon)^2 + 2(x-\varepsilon)z}}{2(x-\varepsilon)}$. So $g(x,z) < \infty$ if and only if $x > \varepsilon$ or $x = 0$. In other words (ii') is true while (ii) is false. Adapting the proof of (i')$\Longrightarrow$(iii) by replacing the set A by $\{X_\theta \in (0,\varepsilon)\}$, which has a nonzero probability if $x \in (0,\varepsilon)$ and θ is not almost surely equal to T_0, we see that we also have (i') but not (i) (so that $V(x,z) < \infty$ if and only if $x \geq \varepsilon$ or $x = 0$).

Remark 2.4 From the previous proof, we also observe that we have $g = +\infty$ everywhere except for $x = 0$ implies $V = +\infty$ everywhere except for $x = 0$. This statement does not require Condition (2.16).

We conclude this section by considering the linear case, which turns out to be degenerate.

Proposition 2.2 *Assume that $\alpha \geq 0$ and let $\ell(x) = x$. Then $V = g$.*

Proof. Observe that:

$$V(x,z) = \mathbb{E}_{x,z}[Z_{T_0}] - W(x) \quad \text{where} \quad W(x) := \sup_{\theta \in \mathcal{T}_0} \mathbb{E}_x X_\theta.$$

Since $\alpha \geq 0$, $X_{t \wedge T_0}$ is a local supermartingale, bounded from below. By Fatou's lemma, this implies that $\mathbb{E}_x X_\theta \leq x$ for $\theta \leq T_0$. $\square$

3 A verification result

Our general approach to solve the optimal detection problem is to exhibit a candidate solution for the dynamic programming equation corresponding to the optimal stopping problem (2.10) which is:

$$\min\{Lv, g - v\} = 0, \text{ on } \mathrm{Int}(\boldsymbol{\Delta}) \tag{3.1}$$

$$v(0,z) = \ell(z) \tag{3.2}$$

$$v_z(z,z) = 0, \tag{3.3}$$

where L is the second order differential operator

$$Lv(x) = v''(x) - \alpha(x)v'(x), \tag{3.4}$$

and α is defined in (2.2). Notice that $LS = 0$. We do not intend to prove directly that V satisfies this differential equation. Instead, we shall guess a candidate solution v of (3.1), and show that v indeed coincides with the value function V by a verification argument.

From now on, we will assume that one of the equivalent relations of (2.14) is satisfied, so that g and V are finite everywhere.

In order to exhibit a solution of (3.1), we guess that there should exist a free boundary $\gamma(x)$ so that stopping is optimal in the region $\{z \geq \gamma(x)\}$, while continuation is optimal in the remaining region $\{z < \gamma(x)\}$. If such a stopping boundary exists, then the above dynamic programming equation reduces to:

$$Lv(x,z) = 0 \text{ for } 0 < z < \gamma(x) \tag{3.5}$$
$$v(x,z) = g(x,z) \text{ and } Lg(x,z) \geq 0 \text{ for } z \geq \gamma(x) \tag{3.6}$$
$$v(0,z) = \ell(z) \tag{3.7}$$
$$v_z(z,z) = 0. \tag{3.8}$$

The verification step requires that the value function be C^1 and piecewise C^2 in order to allow for the application of Itô's formula. We then complement the above system by the continuity and the smoothfit conditions

$$v(x,\gamma(x)) = g(x,\gamma(x)) \tag{3.9}$$
$$v_x(x,\gamma(x)) = g_x(x,\gamma(x)) \tag{3.10}$$

Our objective is to find a candidate v which satisfies (3.5) to (3.10) and an optimal stopping boundary γ so as to apply the following verification result:

Theorem 3.1 *Let γ be continuous and let v be a solution of* (3.5) *to* (3.10)*, which is $C^{1,0}$ and piecewise $C^{2,1}$ w.r.t. (x,z) on $\mathbf{\Delta}$, bounded from below, such that $v \leq g$ on $\mathbf{\Delta}$ and $v < g$ on the continuation region $\{(x,z);\ 0 < x \leq z \text{ and } z < \gamma(x)\}$.*
Then $v = V$ and $\theta^* = \inf\{t \geq 0; Z_t \geq \gamma(X_t)\}$ is an optimal stopping time.
Moreover if τ is another optimal stopping time, then $\theta^* \leq \tau$ a.s.

Proof.
(i) We first prove that $V \geq v$:
Let $\theta \in \mathcal{T}_0$ and for $n \in \mathbb{N}$, define $\theta_n = n \wedge \theta \wedge \inf\{t \geq 0;\ |Z_t| \geq n\}$. Then from the assumed regulrity of v, we may apply Itô's formula to obtain:

$$v(x,z) = v(X_{\theta_n}, Z_{\theta_n}) - \int_0^{\theta_n} Lv(X_t,Z_t)dt - \int_0^{\theta_n} v_x(X_t,Z_t)\sigma(X_t)dW_t - \int_0^{\theta_n} v_z(X_t,Z_t)dZ_t$$

Taking expectations and using the fact that $v_z(X_t,Z_t)dZ_t = v_z(Z_t,Z_t)dZ_t = 0$, $Lv \geq 0$ and $v \leq g$:

$$\begin{aligned} v(x,z) &\leq \mathbb{E}_{x,z} v(X_{\theta_n}, Z_{\theta_n}) \\ &\leq \mathbb{E}_{x,z} g(X_{\theta_n}, Z_{\theta_n}) = \mathbb{E}_{x,z}[\mathbb{E}_{X_{\theta_n},Z_{\theta_n}} \ell(Z_{T_0} - X_{\theta_n})] = \mathbb{E}_{x,z}\ell(Z_{T_0} - X_{\theta_n}). \end{aligned} \tag{3.11}$$

Clearly as $n \to \infty$, $\theta_n \to \theta$ a.s. Notice that $(0 \leq)\ell(Z_{T_0} - X_{\theta_n}) \leq \ell(Z_{T_0})$, which is integrable by (2.14). Therefore, using Lebesgue's dominated convergence theorem while sending $n \to \infty$ in (3.11), we get:

$$v(x,z) \leq \mathbb{E}_{x,z}\ell(Z_{T_0} - X_\theta), \text{ for all } \theta \in \mathcal{T}_0,$$

and therefore $v \leq V$.

(ii) We next prove that $V \leq v$:

If $z \geq \gamma(x)$, then $v = g \geq V$.

Assume now that $z < \gamma(x)$. Let $\theta^* = \inf\{t \geq 0;\ Z_t \geq \gamma(X_t)\}$. By the assumed regularity on v, we have $Lv(X_t, Z_t) = 0$ for $t \in [0, \theta^*)$. As before define $\theta_n = n \wedge \theta^* \wedge \inf\{t \geq 0;\ |Z_t| \geq n\}$, then it follows from Itô's formula that:

$$v(x,z) = \mathbb{E}_{x,z} v(X_{\theta_n}, Z_{\theta_n})$$

Since v is bounded from below and $v \leq g$, we have $|v| \leq c + g$ for some constant c. Since $(0 \leq)\ell(Z_{T_0} - X_{\theta_n}) \leq \ell(Z_{T_0})$, which is integrable by (2.14), $(\mathbb{E}[\ell(Z_{T_0})|X_{\theta_n}, Z_{\theta_n}])_n$ is uniformly integrable, so $(g(X_{\theta_n}, Z_{\theta_n}))_n$ is also uniformly integrable and therefore $(v(X_{\theta_n}, Z_{\theta_n}))_n$ is uniformly integrable too.

So we can claim the following:

$$\begin{aligned} v(x,z) = \mathbb{E}_{x,z} v(X_{\theta^*}, Z_{\theta^*}) = \mathbb{E}_{x,z} v(X_{\theta^*}, \gamma(X_{\theta^*})) &= \mathbb{E}_{x,z} g(X_{\theta^*}, \gamma(X_{\theta^*})) \\ &= \mathbb{E}_{x,z} \ell(Z_{T_0} - X_{\theta^*}) \geq V(x,z). \end{aligned}$$

(iii) Finally we show the minimality of θ^*. Assume to the contrary that there exists τ satisfying $\mathbb{P}(\tau < \theta^*) > 0$ and $\mathbb{E}_{x,z}\ell(Z_{T_0} - X_\tau) = \inf_\theta \mathbb{E}_{x,z}\ell(Z_{T_0} - X_\theta) = V(x,z)$.

But on $\{\tau < \theta^*\}$, we have by assumption $V(X_\tau, Z_\tau) < g(X_\tau, Z_\tau)$, while we always have $V(X_\tau, Z_\tau) \leq g(X_\tau, Z_\tau)$. This leads to the following contradiction:

$$V(x,z) = \mathbb{E}_{x,z}\ell(Z_{T_0} - X_\tau) = \mathbb{E}_{x,z} g(X_\tau, Z_\tau) > \mathbb{E}_{x,z} V(X_\tau, Z_\tau) \geq V(x,z),$$

where the last inequality follows immediately from the definition of V. $\square$

In the rest of this paper, our objective is to exhibit functions γ and v satisfying the assumptions of the previous theorem. In view of (3.6), the stopping region satisfies

$$\{(x,z):\ z \geq \gamma(x)\} \quad \subset \quad \Gamma^+ := \{(x,z):\ Lg(x,z) \geq 0\}\,. \tag{3.12}$$

We therefore need to study the structure of the set Γ^+.

In the subsequent paragraphs we shall first focus on quadratic loss functions. For general loss functions, we shall provide some conditions which guarantee that the structure of the solution agrees with that of the quadratic case.

4 The set Γ^+ for a quadratic loss function

Throughout this section as well as sections 5 and 6, we consider the quadratic loss function

$$\ell(x) := \frac{1}{2}\, x^2 \quad \text{for} \quad x \geq 0,$$

and we assume that the coefficient α satisfies the following additional condition:

$$\text{either } \exists K \geq 0, \text{ for } x \geq K,\ \alpha'(x) = 0, \text{ or, as } x \to \infty,\ \alpha''(x) = \circ\left([\alpha^2]'(x)\right). \tag{4.1}$$

In order to study the set Γ^+ defined by (3.12), we compute that:

$$Lg(x,z) = 1 + \alpha(x)(z-x) - \big(2S'(x) - \alpha(x)S(x)\big)\int_z^\infty \frac{du}{S(u)}, \qquad (x,z)\in\mathbf{\Delta}, \quad (4.2)$$

which takes values in $\mathbb{R}\cup\{-\infty\}$. Since $\alpha\geq 0$ and $2S'-\alpha S\geq 2$ by Remark 2.1, it follows that for every fixed $x\geq 0$, the function $z\longmapsto Lg(x,z)$ is strictly increasing on $[x,\infty)$. Now since $\int_z^\infty \frac{du}{S(u)}\to 0$ when $z\to\infty$, we see that $\lim_{z\to\infty} Lg(x,z)>0$ for any $x\geq 0$. This shows that $\Gamma^+\neq\emptyset$ and that $\Gamma^+ = \mathrm{Epi}(\Gamma) := \{(x,z)\in\mathbf{\Delta};\ z\geq\Gamma(x)\}$ where

$$\Gamma(x) \quad := \quad \inf\{z\geq x:\ Lg(x,z)\geq 0\}, \quad (4.3)$$

Moreover, $\Gamma^+\setminus graph(\Gamma) = \mathrm{Int}(\Gamma^+)\subset\{(x,z)\in\mathbf{\Delta};\ Lg(x,z)>0\}$ and Γ is continuous.

Denote:

$$\Gamma^0 := \Gamma(0) \text{ and } \Gamma^\infty := \sup\{x>0,\ Lg(x,x)<0\}\in(0,+\infty]. \quad (4.4)$$

We also directly compute that for $x>0$:

$$\frac{\partial^2}{\partial x^2}Lg(x,z) \quad = \quad -2\alpha'(x) + \alpha''(x)(z-x) - (\alpha^2(x)S'(x) - \alpha''(x)S(x))\int_z^\infty \frac{du}{S(u)} \ < \ 0$$

by the concavity, the non-decrease, and the positivity of α on $(0,\infty)$. This implies that the function Γ is U-shaped in the sense of Proposition 4.2-(i) below.

We first isolate some asymptotic results that will be needed.

Proposition 4.1 *Under Conditions* (2.2)*, we have the following asymptotic behaviors, as* $z\to\infty$*:*
(i) $S(z)\sim\dfrac{S'(z)}{\alpha(z)}$*;*
(ii) $\displaystyle\int_z^\infty \frac{du}{S(u)}\sim\frac{1}{S'(z)}$, $\displaystyle\int_z^\infty\frac{u}{S(u)}du\sim\frac{z}{S'(z)}$ *and* $\displaystyle\int_z^\infty\frac{u-z}{S(u)}\sim\frac{1}{\alpha(z)S'(z)}$.

Proof. See Appendix. □

Proposition 4.2 *Under Conditions* (2.2)*, we have:* (i) $\Gamma^0>0$ *and there is a constant* $\zeta\geq 0$ *such that* Γ *is decreasing on* $[0,\zeta]$ *and increasing on* $[\zeta,+\infty)$*;*
(ii) $\lim_{x\to+\infty}\Gamma(x)-x=0$*;*
(iii) $0<\Gamma^0<\Gamma^\infty$*, where* Γ^0 *and* Γ^∞ *were defined in* (4.4)*.*

Proof. (i): We first show that for $x_1<x_3$, $\lambda\in(0,1)$ and $x_2=\lambda x_1+(1-\lambda)x_3$, we have $\Gamma(x_2)<\max(\Gamma(x_1),\Gamma(x_3))$.

Indeed, assuming to the contrary that $\Gamma(x_2)\geq\max(\Gamma(x_1),\Gamma(x_3))$, it follows from the strict concavity of Lg w.r.t. x and its non-decrease w.r.t. z that:

$$\begin{aligned} Lg(x_2,\Gamma(x_2)) &> \lambda Lg(x_1,\Gamma(x_2)) + (1-\lambda)Lg(x_3,\Gamma(x_2)) \\ &\geq \lambda Lg(x_1,\Gamma(x_1)) + (1-\lambda)Lg(x_3,\Gamma(x_3)) \geq 0 \end{aligned}$$

By continuity of Lg, $Lg(x_2, \Gamma(x_2)) > 0$ implies that $\Gamma(x_2) = x_2$, which is in contradiction with $\Gamma(x_2) \geq \Gamma(x_3) \geq x_3 > x_2$.

Since $\Gamma(x) \geq x$, this implies the second part of (i).

Finally, we show that $\Gamma^0 = \Gamma(0) > 0$. Since $S(x) \sim x$ as $x \to 0$, we have $\int_0^\infty \frac{du}{S(u)} = \infty$, and therefore $Lg(x,z) < 0$ on $\mathbf{\Delta}$ for z sufficiently small. In particular for sufficiently small $z > 0$, we have $Lg(0,z) < 0$. Then $\Gamma^0 > 0$ and by continuity of Lg, $Lg(0, \Gamma^0) = 0$.

(ii): For an arbitrary $a > 0$, it follows from Proposition 4.1 that:

$$\begin{aligned} Lg(z-a, z) &= 1 + a\alpha(z-a) - \frac{S'(z-a)}{S'(z)} + \circ(1) \\ &= 1 + a\alpha(z-a) - e^{-\int_{z-a}^{z} \alpha(u)du} + \circ(1) \end{aligned}$$

where $\circ(1) \to 0$ as $z \to \infty$.

If $\lim_{x\to\infty} \alpha(x) = +\infty$, then $\frac{S'(z-a)}{S'(z)} \to 0$ (as $z \to \infty$).

If $\lim_{x\to\infty} \alpha(x) = M > 0$ then $\frac{S'(z-a)}{S'(z)} \sim e^{-aM} < 1$.

In both cases, $Lg(z-a, z) > 0$ for z large enough, and so $0 \leq \Gamma(z) - z < a$.

(iii): It remains to prove that $\Gamma^0 < \Gamma^\infty$.

Using Remark 2.1 (ii) and the fact that $Lg(0, \Gamma^0) = 0$, we compute:

$$\begin{aligned} Lg(\Gamma^0, \Gamma^0) &= 1 - (2S' - \alpha S)(\Gamma^0) \int_{\Gamma^0}^\infty \frac{du}{S(u)} \\ &< 1 - 2\int_{\Gamma^0}^\infty \frac{du}{S(u)} \\ &= Lg(0, \Gamma^0) - \alpha(0)\Gamma^0 \\ &= -\alpha(0)\Gamma^0 \leq 0. \end{aligned}$$

By continuity of Lg, this implies that $\Gamma^\infty > \Gamma^0$. □

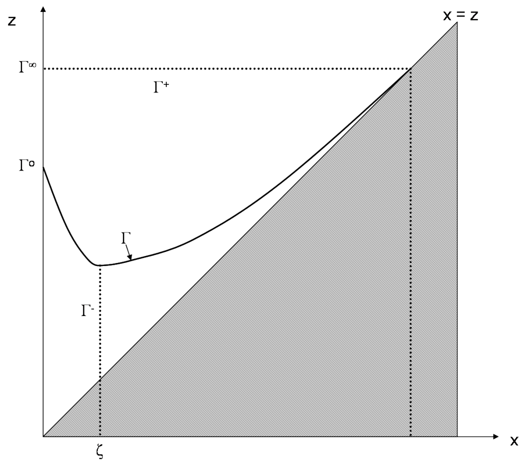
Remark 4.1 If $z \leq \Gamma^0$, then $Lg(0,z) \leq 0$, therefore adapting the proof of Proposition 4.2-(iii), we see that $Lg(z,z) < 0$ for $z \leq \Gamma^0$.

Remark 4.2 The fact $\Gamma^0 < \Gamma^\infty$ implies that, in the quadratic case, the increasing part of Γ will never be reduced to a subset of the diagonal, or in other words that $\Gamma(\zeta) > \zeta$.

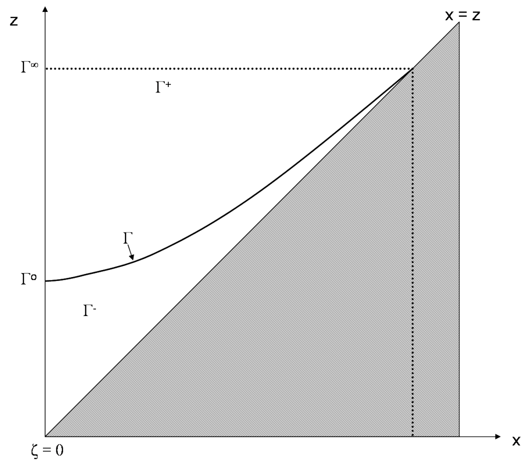
The figures below exhibit the two possible shapes of the function Γ and the location of Γ^+. Notice that in both cases Γ^∞ can be finite or not. We refer the reader to section 7 for examples of both cases.

We now give a result stronger than Proposition 4.2-(ii) above, concerning the behavior of Γ at infinity. Recall that Γ^∞ was defined by (4.4).

Proposition 4.3 *Let the coefficient α satisfy Conditions* (2.2) *and* (4.1). *Then, there exists $\Gamma^{\max} > 0$ such that:*
- either for any $x \geq \Gamma^{\max}$, $\Gamma(x) > x$
- or for any $x \geq \Gamma^{\max}$, $\Gamma(x) = x$.
Moreover, if $\lim_{x\to\infty} \alpha(x) = \infty$, then $\Gamma^\infty < \infty$.



(a) $\zeta > 0$ (Ornstein-Uhlenbeck process)



(b) $\zeta = 0$ (Brownian motion with negative drift)

Figure 1: The two possible shapes of Γ

Proof. By the definition of the scale function (2.3), for $x > 0$:

$$S(x) = S(1) + \frac{S'(x)}{\alpha(x)} - \frac{S'(1)}{\alpha(1)} - \int_1^x \left(\frac{1}{\alpha}\right)'(u)S'(u)du. \tag{4.5}$$

We then consider several cases:

Case 1: $\int_1^\infty (1/\alpha)'(u)S'(u)du > -\infty$. Then $S(x) = A(x) + \frac{S'(x)}{\alpha(x)}$ for some non-decreasing function A which is bounded on $[1, \infty)$. We write $A^\infty := \lim_{x\to\infty} A(x)$.

Case 1.1: If there exists $K \geq 0$ such that $A(x) = 0$ for any $x \geq K$, then, recalling that $LS = 0$, we compute for $x \geq K$:

$$\int_x^\infty \frac{du}{S(u)} = \int_x^\infty \frac{\alpha(u)du}{S'(u)} = \frac{1}{S'(x)},$$

and then:

$$\begin{aligned} Lg(x,x) &= 1 - [2S'(x) - \alpha(x)S(x)] \int_x^\infty \frac{du}{S(u)} \\ &= 1 - S'(x)\frac{1}{S'(x)} = 0. \end{aligned}$$

Consequently, for $x \geq K$, $\Gamma(x) = x$.

Case 1.2: If on the other hand, $A^\infty \neq 0$ or $A^\infty = 0$ but $\alpha'(x) > 0$ for any $x \geq 0$, then notice that if $A^\infty \leq 0$, then $A(x) < 0$ for any x, and if $A^\infty > 0$, then $A(x) > 0$ for sufficiently large x.

Let us prove that $A'(x) \to 0$ as $x \to \infty$. By definition: $A'(x) = \frac{\alpha'(x)S'(x)}{\alpha(x)^2}$. If $\alpha'(x) = 0$ for sufficiently large x, the conclusion is immediate. Otherwise, we compute:

$$A''(x) = \frac{\alpha''(x)\alpha(x) + \alpha'(x)\alpha(x)^2 - 2\big(\alpha'(x)\big)^2}{\alpha(x)^3} S'(x).$$

Using the fact that $\alpha' = \circ(\alpha^2)$ and condition (4.1), we see that $A''(x) \sim \frac{\alpha'(x)S'(x)}{\alpha(x)} > 0$, as $x \to \infty$. Therefore A' is increasing for sufficiently large x, and as A is bounded, $A'(x) \to 0$.

Recalling that $LS = 0$, we compute:

$$\begin{aligned}\int_x^\infty \frac{du}{S(u)} &= \int_x^\infty \frac{du}{A(u) + \frac{S'(u)}{\alpha(u)}} = \int_x^\infty \frac{\alpha(u)}{S'(u)} \frac{du}{1 + A(u)\frac{\alpha(u)}{S'(u)}} \\ &= \frac{1}{S'(x)} - \int_x^\infty A(u) \frac{\alpha^2(u)}{[S'(u)]^2} du + \circ\left(\int_x^\infty A(u)\frac{\alpha^2(u)}{[S'(u)]^2}\right).\end{aligned}$$

Integrating by part:

$$\int_x^\infty A(u)\frac{\alpha^2(u)}{[S'(u)]^2} du = \frac{A(x)\alpha(x)}{[S'(x)]^2} + \int_x^\infty \frac{\alpha'(u)A(u) + \alpha(u)A'(u)}{[S'(u)]^2} - \int_x^\infty A(u)\frac{\alpha^2(u)}{[S'(u)]^2} du,$$

so that:

$$\int_x^\infty \frac{du}{S(u)} = \frac{1}{S'(x)}\left[1 - \frac{A(x)\alpha(x)}{2S'(x)} + \circ\left(\frac{A(x)\alpha(x)}{S'(x)}\right)\right]$$

where we used the fact that $(1/\alpha)' \to 0$, see Remark 2.1, and that $A'(x) \to 0$. Then:

$$\begin{aligned}Lg(x,x) &= 1 - [2S'(x) - \alpha(x)S(x)]\int_x^\infty \frac{du}{S(u)} \\ &= 1 - (S'(x) - A(x)\alpha(x))\frac{1 - \frac{A(x)\alpha(x)}{2S'(x)} + \circ\left(\frac{A(x)\alpha(x)}{S'(x)}\right)}{S'(x)} \\ &= \frac{3A(x)\alpha(x)}{2S'(x)} + \circ\left(\frac{A(x)\alpha(x)}{S'(x)}\right)\end{aligned}$$

Consequently, for sufficiently large x, if $A^\infty > 0$, $\Gamma(x) = x$, while if $A^\infty \leq 0$, $\Gamma(x) > x$.

Case 2: $\int_1^\infty \left(\frac{1}{\alpha}\right)'(u)S'(u)du = -\infty$. Then $\alpha'(x) > 0$ for all $x \geq 0$.
We compute that:

$$\int_1^x \left(\frac{1}{\alpha}\right)'(u)S'(u)du = \left[\left(\frac{1}{\alpha}\right)'(u)\frac{S'(u)}{\alpha(u)}\right]_1^x - \int_1^x \left[\frac{1}{\alpha}\left(\frac{1}{\alpha}\right)'\right]'(u)\frac{S'(u)}{\alpha(u)}du.$$

By (4.1), we observe that:

$$\frac{1}{\alpha}\left[\frac{1}{\alpha}\left(\frac{1}{\alpha}\right)'\right]' = \frac{\alpha\alpha'' - (\alpha')^2}{\alpha^5} = \circ\left(\frac{\alpha'}{\alpha^2}\right),$$

and therefore:

$$\int_1^x \left(\frac{1}{\alpha}\right)'(u)S'(u)du = \left(\frac{1}{\alpha}\right)'(x)\frac{S'(x)}{\alpha(x)} + \circ\left(\left(\frac{1}{\alpha}\right)'(x)\frac{S'(x)}{\alpha(x)}\right).$$

Plugging this in (4.5), we see that:

$$S(x) = \frac{S'(x)}{\alpha(x)}\left[1 - \left(\frac{1}{\alpha}\right)'(x) + \circ\left(\left(\frac{1}{\alpha}\right)'(x)\right)\right],$$

which implies that:

$$\int_x^\infty \frac{du}{S(u)} = \frac{1}{S'(x)} + \int_x^\infty \frac{\alpha(u)}{S'(u)} \left(\frac{1}{\alpha}\right)'(u)du + \circ\left(\int_x^\infty \frac{\alpha(u)}{S'(u)} \left(\frac{1}{\alpha}\right)'(u)du\right).$$

Integrating by part and using (4.1), this provides:

$$\begin{aligned}\int_x^\infty \frac{\alpha(u)}{S'(u)} \left(\frac{1}{\alpha}\right)'(u)du &= \left(\frac{1}{\alpha}\right)'(x)\frac{1}{S'(x)} + \int_x^\infty \left(\frac{1}{\alpha}\right)''(u)\frac{du}{S'(u)} \\ &= \left(\frac{1}{\alpha}\right)'(x)\frac{1}{S'(x)} + \circ\left(\left(\frac{1}{\alpha}\right)'(x)\frac{1}{S'(x)}\right).\end{aligned}$$

Hence:

$$\begin{aligned}Lg(x,x) &= 1 - [2S'(x) - \alpha(x)S(x)]\int_x^\infty \frac{du}{S(u)} \\ &= 1 - \left[1 + \left(\frac{1}{\alpha}\right)'(x) + \circ\left(\left(\frac{1}{\alpha}\right)'(x)\right)\right]\left[1 + \left(\frac{1}{\alpha}\right)'(x) + \circ\left(\left(\frac{1}{\alpha}\right)'(x)\right)\right] \\ &= -2\left(\frac{1}{\alpha}\right)'(x) + \circ\left(\left(\frac{1}{\alpha}\right)'(x)\right).\end{aligned}$$

Since $(1/\alpha)' < 0$, this implies that for sufficiently large x, $Lg(x,x) > 0$ and therefore $\Gamma(x) = x$.

It remains to prove that if $\lim_{x\to\infty} \alpha(x) = \infty$, then $\Gamma^\infty < \infty$.
Let $x \geq 1$, since α is non-decreasing, we have:

$$S'(x) = e^{\int_0^x \alpha(u)du} \geq e^{\alpha(x-1)}.$$

Since α' is non-increasing and non-negative, α' is bounded. Therefore, there exists $K > 0$, such that $0 \leq \alpha(x) - \alpha(x-1) \leq K$, and therefore $S'(x) \geq e^{\alpha(x)-K}$ for $x \geq 1$.
On the other hand, $\lim_{x\to\infty} \alpha(x) = \infty$ implies that $\alpha(x)^2 = \circ(e^{\alpha(x)-K})$, which means that $\frac{S'(x)}{\alpha(x)^2} \to \infty$. Finally, as $x \to \infty$, we get:

$$\alpha'(x) = \circ\left(-\left(\frac{1}{\alpha}\right)'(x)S'(x)\right).$$

As α is not bounded, the left-hand side is not integrable at infinity, so the right-hand side is also not integrable. In other words, $\int_1^\infty (1/\alpha)'(u)S'(u)du = -\infty$, and from Case 2 above, we have $\Gamma^\infty < \infty$. □

5 The stopping boundary in the quadratic case

We now turn to the characterization of the stopping boundary γ. Following Proposition 4.2-(i), we define:

$$\Gamma_\downarrow = \Gamma_{|_{[0,\zeta]}} \qquad \text{and} \quad \Gamma_\uparrow = \Gamma_{|_{[\zeta,\infty)}}$$

the restrictions of Γ to the intervals $[0,\zeta]$ and $[\zeta,\infty)$.

5.1 The increasing part of the stopping boundary γ

Our objective is to find a solution v of (3.5)–(3.10) on $\{(x,z)\in\mathbf{\Delta};\ x<z<\gamma(x)\}$. First by (3.5), v is of the form:

$$v(x,z)=A(z)+B(z)S(x).$$

Then, on the interval where γ is one-to-one, the continuity and smoothfit conditions (3.9) and (3.10) imply that

$$v(x,z)=g(\gamma^{-1}(z),z)+\frac{g_x(\gamma^{-1}(z),z)}{S'(\gamma^{-1}(z))}[S(x)-S(\gamma^{-1}(z))].$$

Finally, the Neumann condition (3.8), implies that the boundary γ satisfies the following ODE:

$$\gamma'=\frac{Lg(x,\gamma)}{1-\frac{S(x)}{S(\gamma)}} \tag{5.1}$$

In the sequel, we take this ODE (with no intial condition !) as a starting point to construct the boundary γ. Notice that this ODE has infinitely many solutions, as the Cauchy-Lipschitz condition is locally satisfied whenever (5.1) is complemented with the condition $\gamma(x_0)=z_0$ for any $0<x_0<z_0$. This is a similar feature as in Peskir [11]. The following result selects an appropriate solution of (5.1).

Proposition 5.1 *Let the coefficient α satisfy Conditions* (2.2) *and* (4.1). *Then, there exists a continuous function γ defined on $\mathbb{R}_+$ with graph $\{(x,\gamma(x)) : x\in\mathbb{R}_+\}\subset\mathbf{\Delta}$, such that:*
(i) *on the set $\{x>0:\ \gamma(x)>x\}$, γ is a C^1 solution of the ODE* (5.1),
(ii) *$\{(x,\gamma(x)):\ x\geq\zeta\}\subset\Gamma^+$, and $\{(x,\gamma(x)):\ x>\zeta$ and $\gamma(x)>x\}\subset Int(\Gamma^+)$,*
(iii) *if $\Gamma^\infty<\infty$, then $\gamma(x)=x$ for all $x\geq\Gamma^\infty$,*
(iv) *$\gamma(x)-x\longrightarrow 0$ as $x\to\infty$.*

The remaining part of this section is dedicated to the proof of this result. We first introduce some notations. We recall from Remark 4.2 that the graph of $\Gamma_\uparrow$ is not reduced to the diagonal and therefore

$$b\ :=\ \inf\{x\geq 0:\ \Gamma(x)=x\}\ \in\ (\zeta,\infty] \tag{5.2}$$

where b may take infinite value. We also introduce

$$\mathbf{D}^-\ :=\ \{x>\zeta:\ Lg(x,x)<0\}\supset(\zeta,b), \tag{5.3}$$

and for all $x_0\in\mathbf{D}^-$:

$$d(x_0):=\sup\{x\leq x_0:\ Lg(x,x)\geq 0\}\quad\text{and}\quad u(x_0):=\inf\{x\geq x_0;\ Lg(x,x)\geq 0\}, \tag{5.4}$$

with the convention that $d(x_0)=0$ if $\{x\leq x_0:\ Lg(x,x)\geq 0\}=\emptyset$ and $u(x_0)=\infty$ if $\{x\geq x_0;\ Lg(x,x)\geq 0\}=\emptyset$. Since Lg is continuous and $x_0\in\mathbf{D}^-$, $d(x_0)<x_0<u(x_0)\leq\infty$. Notice that if $x_0\in(\zeta,b)$, then $d(x_0)=0$.

Let $x_0 \in \mathbf{D}^-$ be an arbitrary point. For all $z_0 > x_0$, we denote by $\gamma_{x_0}^{z_0}$ the maximal solution of the Cauchy problem complemented with the condition $\gamma(x_0) = z_0$, and we denote $\left(\ell_{x_0}^{z_0}, r_{x_0}^{z_0}\right)$ the associated (open) interval. When x_0 will be fixed, we will simply denote γ^{z_0}, ℓ^{z_0} and r^{z_0}. Notice that since the right-hand side of ODE (5.1) is locally lipschitz on the set $\{(x,\gamma),\ 0 < x < \gamma\}$, the maximal solution will be defined as long as $0 < x < \gamma$.

The following result provides more properties on the maximal solutions.

Lemma 5.1 *Assume that α satisfies Conditions* (2.2) *and let $x_0 \in \mathbf{D}^-$ be fixed.*
(i) *For all $z > x_0$, $\ell^z \le d(x_0)$ and if $\ell^z > 0$, then $Lg(\ell^z, \ell^z) \ge 0$;*
(ii) *for all $z \in (x_0, \Gamma(x_0)]$, $Lg(x, \gamma^z(x)) < 0$ for any $x \in (x_0, r^z)$;*
(iii) *there exists $D > 0$, such that $r^z = +\infty$ for all $z \ge D$.*

Before proceeding to the proof of this result, we turn to the main construction of the stopping boundary γ. Let

$$\mathbf{Z}(x_0) := \left\{z > x_0 :\ Lg\big(x, \gamma_{x_0}^z(x)\big) < 0 \text{ for some } x \in \left[x_0, r_{x_0}^z\right)\right\},\ z^*(x_0) := \sup \mathbf{Z}(x_0). \quad (5.5)$$

Moreover, whenever $z^*(x_0) < \infty$, we denote

$$\gamma_{x_0}^* := \gamma_{x_0}^{z^*(x_0)},\ \ell_{x_0}^* := \ell_{x_0}^{z^*(x_0)},\ r_{x_0}^* := r_{x_0}^{z^*(x_0)}, \quad \text{and} \quad I_{x_0}^* := \big(\ell_{x_0}^*, r_{x_0}^*\big). \quad (5.6)$$

Lemma 5.2 *Assume that α satisfies Conditions* (2.2) *and let x_0 be arbitrary in $\mathbf{D}^-$. Then*
(i) *$z^*(x_0) \in (\Gamma(x_0), \infty)$, $\big(d(x_0), u(x_0)\big) \subset I_{x_0}^*$, and the corresponding maximal solution $\gamma_{x_0}^*$ has a positive derivative on the interval $I_{x_0}^* \cap (\zeta, \infty)$.*
(ii) *For $x_0 \le x_1 \in \mathbf{D}^-$, we have*
- *either $I_{x_0}^* \cap I_{x_1}^* = \emptyset$,*
- *or $I_{x_0}^* \subset I_{x_1}^*$ and $\gamma_{x_0}^* \le \gamma_{x_1}^*$ on $I_{x_0}^*$.*

Proof. (i) By Lemma 5.1 (iii), there exists $D = D(x_0)$ such that for any $z \ge D$, $r_{x_0}^z = \infty$. If $Lg\big(x_1, \gamma_{x_0}^z(x_1)\big) < 0$ for some $x_1 \ge x_0$, then by (5.1), $\gamma_{x_0}^z$ is decreasing in a neighborhood of x_1 and as long as $\big(x, \gamma_{x_0}^z(x)\big) \in \mathrm{Int}(\Gamma^-)$. Since $x_1 \ge x_0 > \zeta$, Γ is increasing on $[x_1, \infty)$, so that $\gamma_{x_0}^z$ is decreasing on $[x_0, r^z)$, which implies that $r_{x_0}^z \le z$. Therefore $\mathbf{Z}(x_0)$ is bounded by D, and $z^*(x_0) < \infty$. Since $x_0 \in \mathbf{D}^-$, we have $\Gamma(x_0) \in \mathbf{Z}(x_0)$ and therefore $z^*(x_0) \ge \Gamma(x_0)$. We next assume to the contrary that $z^*(x_0) = \Gamma(x_0)$ and work towards a contradiction.

Notice that $\mathbf{D}^-$ is an open set as a consequence of the continuity of the function Lg. Then there exists $\varepsilon > 0$ such that $(x_0, x_0 + 2\varepsilon) \subset \mathbf{D}^- \cap (x_0, r_{x_0}^*)$ and $d(x) = d(x_0)$ for any $x \in (x_0, x_0 + \varepsilon)$. Let $x_\varepsilon := x_0 + \varepsilon$ and $z_\varepsilon := \Gamma(x_\varepsilon) > \Gamma(x_0)$. By Lemma 5.1 (i), we have $\ell_{x_\varepsilon}^{z^\varepsilon} \le d(x_0) < x_0$, and it follows from Lemma 5.1 (ii) that $\gamma_{x_\varepsilon}^{z_\varepsilon}$ is decreasing on $\big(x_0, r_{x_\varepsilon}^{z^\varepsilon}\big)$. Then:

$$\gamma_{x_\varepsilon}^{z_\varepsilon}(x_0) > \gamma_{x_\varepsilon}^{z_\varepsilon}(x_\varepsilon) = \Gamma(x_\varepsilon) > \Gamma(x_0) = z^*. \quad (5.7)$$

On the other hand, since $\gamma_{x_0}^{\gamma_{x_\varepsilon}^{z_\varepsilon}(x_0)}(x_\varepsilon) = z^\varepsilon = \Gamma(x_\varepsilon)$, it follows from Lemma 5.1 (ii) that $\gamma_{x_\varepsilon}^{z^\varepsilon}(x_0) \in \mathbf{Z}(x_0)$, implying that $z^* \ge \gamma_{x_\varepsilon}^{z^\varepsilon}(x_0) \in \mathbf{Z}(x_0)$, a contradiction to (5.7).

A similar argument proves that $(x, \gamma^*(x)) \in \mathrm{Int}(\Gamma^+)$ for $x \in I_{x_0}^* \cap [\zeta, \infty)$, which implies $r_{x_0}^* \ge u(x_0)$ (possibly infinite). Using (5.1), we see that γ has a positive derivative on the same interval. Finally, Lemma 5.1-(i) implies that $\ell_{x_0}^* \le d(x_0)$.

2) Let $x_0 < x_1$ in $\mathbf{D}^-$ and assume that there exists $x_2 \in I^*_{x_0} \cap I^*_{x_1}$. If $\gamma^*_{x_0}(x_2) = \gamma^*_{x_1}(x_2)$, then the one-to-one property of the flow and the maximality of I^* imply that $I^*_{x_0} = I^*_{x_1}$ and $\gamma^*_{x_0} = \gamma^*_{x_1}$. If $\gamma^*_{x_0}(x_2) < \gamma^*_{x_1}(x_2)$, the one-to-one property of the flow implies that $\gamma^*_{x_0} < \gamma^*_{x_1}$ on $I^*_{x_0} \cap I^*_{x_1}$ and therefore the maximality of $I^*_{x_1}$ implies that $I^*_{x_0} \subset I^*_{x_1}$. By the definition of $z^*(x_1)$ and the continuity of the flow with respect to initial data, there exists $z < z^*(x_1)$, such that $\gamma^*_{x_0}(x_2) < \gamma^z_{x_1}(x_2) < \gamma^*_{x_1}(x_2)$ and $z \in \mathbf{Z}(x_1)$. For the same reasons as before, we have $I^*_{x_0} \subset I^z_{x_1}$ and $\gamma^*_{x_0} < \gamma^z_{x_1} < \gamma^*_{x_1}$ on $I^*_{x_0}$. Therefore $\gamma^z_{x_1}(x_0) \in \mathbf{Z}(x_0)$ while $\gamma^z_{x_1}(x_0) > z^*(x_0) = \gamma^*_{x_0}(x_0)$, which is impossible. A similar argument can be used if $\gamma^*_{x_0}(x_2) > \gamma^*_{x_1}(x_2)$. □

We are now ready for:

Proof of Proposition 5.1 We first define γ and then prove the announced properties.
1. Let

$$\mathcal{D} := \bigcup_{x_0 \in \mathbf{D}^-} I^*(x_0) \quad \supset \quad \mathbf{D}^-. \tag{5.8}$$

By Lemma 5.2, for any x and y in $\mathbf{D}^-$, we either have $I^*_x = I^*_y$ or $I^*_x \cap I^*_y = \emptyset$. Therefore, assuming the axiom of choice, there exists a subset $\mathbf{D}^-_0$ of $\mathbf{D}^-$ such that $\mathcal{D} = \bigcup_{x_0 \in \mathbf{D}^-_0} I^*(x_0)$ and for any $x, y \in \mathbf{D}^-_0$, $x \neq y$ implies that $I^*_x \cap I^*_y = \emptyset$.

We now define the function γ on $\mathbb{R}_+ \setminus \{0\}$ by:

$$\gamma(x) := \begin{cases} \gamma^*_{x_0}(x) \text{ if } x \in I^*_{x_0}, \text{ for some } x_0 \in \mathbf{D}^- \\ x \text{ otherwise.} \end{cases} \tag{5.9}$$

By Lemma 5.2, this definition does not depend on the choice of $\mathbf{D}^-$.
2. We first prove that γ is continuous on $\mathbb{R}_+$. This is only nontrivial at the endpoints $\ell^*_{x_0}$ and $r^*_{x_0}$, $x_0 \in \mathbf{D}^-$. Recalling that γ is increasing on $I^*_{x_0}$, we see that both limits exist. By the maximality of $I^*(x_0)$, it is immediate that $\lim_{r^*_{x_0}} \gamma = r^*_{x_0}$ and, whenever $\ell^*_{x_0} > 0$, $\lim_{\ell^*_{x_0}} \gamma = \ell^*_{x_0}$. If $\ell^*_{x_0} = 0$, which is the case for $x_0 \in (\zeta, b)$, then the limit also exists and in fact is positive since $Lg(x, \gamma(x)) < 0$ for any $x > 0$ such that $\gamma(x) < \zeta$. Setting $\gamma(0) := \lim_{x \to 0} \gamma(x)$, we obtain a continuous function γ on $\mathbb{R}_+$.
3. Proposition 5.1 (i) follows immediately from Lemma 5.2. To prove (ii), we first notice that $\{x \geq \zeta : \ \gamma(x) = x\} = \mathbb{R}_+ \setminus \mathcal{D} \subset \mathbb{R}_+ \setminus \mathbf{D}^-$, so that $Lg(x, x) \geq 0$ on the set $\{x \geq \zeta : \ \gamma(x) = x\}$. On the set $\{x > \zeta : \ \gamma(x) > x\}$, since γ has a positive derivative and satisfies (5.1), we have $Lg(x, \gamma(x)) > 0$. Finally, since for $x_0 \in (\zeta, b)$, where b was defined by (5.2), $d(x_0) = 0$, Lemma 5.2 and the continuity of Lg imply that $Lg(\zeta, \gamma(\zeta)) \geq 0$.
4. We next prove (iii). Assume $\Gamma^\infty < \infty$ and let $x_0 \in \mathbf{D}^-$ be arbitrary. Then by continuity of Lg, $Lg(\Gamma^\infty, \Gamma^\infty) = 0$, and therefore $x_0 < \Gamma^\infty$. Assume that $r^*_{x_0} > \Gamma^\infty$, and let us work towards a contradiction. Then by continuity of the flow with respect to the initial data, there exists $\varepsilon > 0$ such that for any $z \in \big(z^*(x_0) - \varepsilon, z^*(x_0)\big)$, the function $\gamma^z_{x_0}$ is defined on $[x_0, \frac{\Gamma^\infty + r^*_{x_0}}{2}]$. By Lemma 5.1 (ii), we deduce that $(x, \gamma^z_{x_0}(x)) \in \Gamma^+$ on the same interval. By the definition of Γ^∞ and recalling that $\frac{\partial}{\partial z} Lg > 0$, we get that $z \notin \mathbf{Z}(x_0)$. By the arbitraryness of z in $(z^*(x_0) - \varepsilon, z^*(x_0))$, this contradicts the definition of $z^*(x_0)$.

5. We finally prove (iv). First, the claim is obvious when $\mathcal{D}$ is bounded, as $\gamma(x) = x$ for $x \geq \sup \mathcal{D}$. We then concentrate on the case where $\mathcal{D}$ is not bounded. From Proposition 4.3, $\mathbf{D}^-$ is either bounded or $Lg(x, x) < 0$ for any $x \in [\Gamma^{\max}, \infty)$, and by Lemma 5.2, $r^*_{x_0} \geq u(x_0)$. In both cases, there exists $x_0 \in \mathbf{D}^-$ such that $r^*_{x_0} = \infty$. To complete the proof, we now intend to show that, for $a > 0$ and $x > x_0$ large enough, $\gamma(x) \leq x + a$.

Using Proposition 4.1, we compute:

$$Lg(x, x + a) = 1 + a\alpha(x) - e^{-\int_x^{x+a} \alpha(u)du} + \circ(1). \tag{5.10}$$

- If $\lim_{x\to\infty} \alpha(x) = \infty$, then, for any $\varepsilon > 0$, we get that $Lg(x, x + a) > 1 + \varepsilon$ for x large enough.
- If $\lim_{x\to\infty} \alpha(x) = M > 0$, then $\frac{Lg(x,x+a)}{1-\frac{S(x)}{S(x+a)}} = \frac{1-e^{-aM}+aM}{1-e^{-aM}} + \circ(1)$, so that for any $\varepsilon \in \big(0, \frac{aM}{1-e^{-aM}}\big)$, we get that $\frac{Lg(x,x+a)}{1-\frac{S(x)}{S(x+a)}} > 1 + \varepsilon$ for x large enough.

In both cases, we can find a sufficiently small $\varepsilon > 0$, such that $\frac{Lg(x,x+a)}{1-\frac{S(x)}{S(x+a)}} > 1 + \varepsilon$ for any sufficiently large x, say $x \geq x_1$. We now assume that $\gamma(x_1) > x_1 + a$ and work towards a contradiction. Since $\gamma(x) > x$ on $[x_0, +\infty)$, using the continuity of the flow with respect to the initial data, we can find $z < z^*(x_0)$ such that $\gamma^z_{x_0}(x) > x$ on $[x_0, x_1]$ and $\gamma^z_{x_0}(x_1) > x_1+a$. Using (5.10) together with (5.1), we therefore have for $x \in [x_1, +\infty)$:

$$\begin{aligned}
\gamma^z_{x_0}(x) - \gamma^z_{x_0}(x_1) &\geq (1 + \varepsilon)(x - x_1) \\
\text{and so } \gamma^z_{x_0}(x) &> (1 + \varepsilon)(x - x_1) + x_1 + a \geq x + a,
\end{aligned}$$

so that $r^z = \infty$ and the same holds for any $y \in [z, z^*(x_0)]$, which contradicts the definition of $z^*(x_0)$ as $\sup \mathbf{Z}(x_0)$. □

We finally turn to the proof of Lemma 5.1. Let

$$\Gamma^- := \{(x, z) \in \mathbf{\Delta} : \ Lg(x, z) \leq 0\}. \tag{5.11}$$

Proof of Lemma 5.1 (i) The right-hand side of (5.1) is locally Lipschitz as long as $0 < x < \gamma^z_{x_0}(x)$. Now $\gamma^z_{x_0}$ is non-increasing if $(x, \gamma^z_{x_0}(x)) \in \Gamma^-$. Therefore, since $d(x_0) < x_0 < u(x_0)$ and $\Gamma(x) > x$ for any $x \in \mathbf{D}^- \supset \big(d(x_0), u(x_0)\big)$, the minimality of $\ell^z_{x_0}$ implies that $\ell^z_{x_0} \leq d(x_0)$ and that $\ell^z_{x_0} \notin \mathbf{D}^-$.
(ii) Since $x_0 > \zeta$, the function Γ is increasing on $[x_0, +\infty)$, while by (5.1), for any $z > x_0$, $\gamma^z_{x_0}$ is non-increasing as long as $(x, \gamma_{x_0}(x)) \in \Gamma^-$. Therefore for any $z \in (x_0, \Gamma(x_0))$, $(x, \gamma^z_{x_0}(x))$ remains in $\mathrm{Int}(\Gamma^-)$ on $\big[x_0, r^z_{x_0}\big)$.

Assume now that $z = \Gamma(x_0)$. Since $\Gamma(x_0) > x_0$, Γ satisfies $Lg(x, \Gamma(x)) = 0$ in a neighborhood of x_0. Since $\frac{\partial}{\partial z} Lg > 0$ on $\mathbf{\Delta}$, while $\frac{\partial}{\partial x} Lg(x, \Gamma(x)) > 0$ as soon as $\Gamma(x) > x$, the implicit functions theorem implies that Γ is C^1 with positive derivative in a neighborhood of x_0. If $z = \Gamma(x_0)$, $(\gamma^z_{x_0})'(x_0) = 0$ by (5.1), therefore $\gamma' - \Gamma'$ is negative in a neighborhood of x_0, and we can conclude as in the case $z < \Gamma(x_0)$ that $\big(x, \gamma^z_{x_0}(x)\big) \in \mathrm{Int}(\Gamma^-)$ on $\big(x_0, r^z_{x_0}\big)$.
(iii) Let $\varepsilon > 0$ be given. From Proposition 4.1-(ii), we see that:

$$\begin{aligned}
Lg(x, (1 + \varepsilon)x) &= 1 + \varepsilon x\alpha(x) - \frac{S'(x)}{S'((1 + \varepsilon)x)} + \circ(1) \\
&= 1 + \varepsilon x\alpha(x) - e^{-\int_x^{x+\varepsilon x} \alpha(v)dv} + \circ(1), \quad \text{as } x \to \infty.
\end{aligned}$$

Since $x\alpha(x) \to +\infty$, this implies that:

$$Lg(.,(1+\varepsilon).) \geq 1+3\varepsilon, \text{ on } [A,\infty) \quad \text{for some} \quad A \geq 0. \tag{5.12}$$

In particular, $(A,(1+\varepsilon)A) \in \mathrm{Int}(\Gamma^+)$. Let $D := \max((1+\varepsilon)A, \Gamma^0)$. Since Γ is U-shaped, it follows that $[0,A] \times [D,\infty) \subset \Gamma^+$. Since $\gamma^z_{x_0}$ is non-decreasing as long as $(x, \gamma^z_{x_0}(x)) \in \mathrm{Int}(\Gamma^+)$, by (5.1), it follows that $r^z_{x_0} > A$ and $\gamma^z_{x_0}(A) > (1+\varepsilon)A$ for all $z \geq D$.

In order to complete the proof, we now show that:

$$\gamma^z_{x_0}(x) \geq (1+\varepsilon)x, \text{ for all } x \geq A \text{ and } z \geq D.$$

To see this, assume to the contrary that $\gamma^z_{x_0}(\xi) \leq (1+\varepsilon)\xi$ for some $\xi > A$ and define:

$$x_1 := \inf\{x > A;\ \gamma^z_{x_0}(x) = (1+\varepsilon)x)\}.$$

By continuity of $\gamma^z_{x_0}$, we have $A < x_1 \leq \xi$, and therefore $Lg(x_1,(1+\varepsilon)x_1) \geq 1+3\varepsilon$ by (5.12). Since Lg is also continuous, there is a neighborhood $\mathcal{O}$ of $(x_1,(1+\varepsilon)x_1)$ such that for $(x,z) \in \mathcal{O}$, $Lg(x,z) \geq 1+2\varepsilon$. We then deduce that there exists $\eta > 0$ such that:

$$(\gamma^z_{x_0})'(x) \geq Lg(x, \gamma^z(x)) \geq 1+2\varepsilon \quad \text{for any} \quad x \in [x_1-\eta, x_1+\eta],$$

and then, for $x \in (x_1-\eta, x_1) \cap [A,\infty)$:

$$\gamma^z_{x_0}(x) \leq \gamma^z_{x_0}(x_1) - (1+2\varepsilon)(x_1-x) = (1+\varepsilon)x_1 - (1+2\varepsilon)(x_1-x) < (1+\varepsilon)x.$$

Since $\gamma^z_{x_0}(A) > (1+\varepsilon)A$, this contradicts the definition of x_1. □

5.2 The decreasing part

The problem now is that there is no reason for the function γ constructed in the previous paragraph to be entirely in Γ^+ as it can cross graph$(\Gamma_\downarrow)$. In Section 7, numerical computations suggest that this is indeed the case in the context of an Ornstein-Uhlenbeck process. In fact, the boundary is in general made of two parts as shown on the plot below. Therefore we need to consider the area that lies between the axis $\{x = 0\}$ and graph(γ). While the right part of γ is characterized by the ODE of the previous paragraph because of the Neumann condition, here we must take into account the Dirichlet condition (3.7).

Therefore, we consider the following problem, for a fixed $z > 0$:

$$f\big(x(z), z\big) = 0, \text{ where } f(x,z) = g(x,z) - g_x(x,z)\frac{S(x)}{S'(x)} - \frac{z^2}{2}. \tag{5.13}$$

Proposition 5.2 *Assume that α satisfies Conditions (2.2) and that $\Gamma_\downarrow$ is not degenerate (i.e. $\zeta > 0$). Then there exists $x^* > 0$ and a function $\gamma_\downarrow$ defined on $[0,x^*]$, which is C^0 on $[0,x^*]$, C^1 with negative derivative on $(0,x^*)$ and such that:*
(i) $\forall x \in [0,x^*],\ f(x,\gamma_\downarrow(x)) = 0$
(ii) $\forall x \in (0,x^*),\ (x,\gamma_\downarrow(x)) \in Int(\Gamma^+)$
(iii) $\gamma_\downarrow(0) = \Gamma^0$
(iv) $(x^*, \gamma_\downarrow(x^*)) \in graph(\Gamma_\uparrow)$

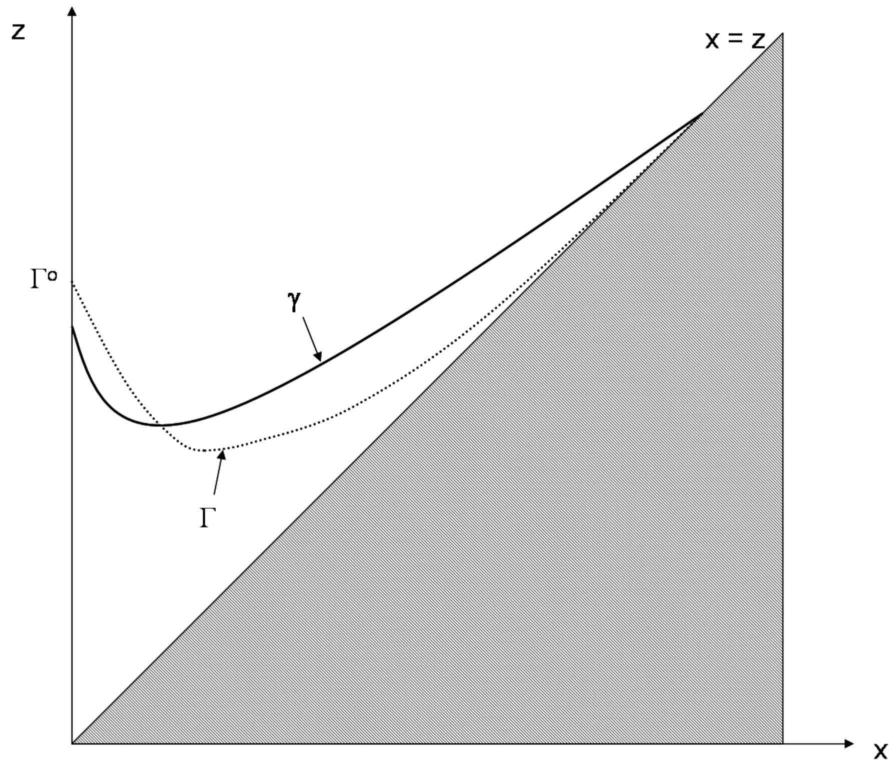


Figure 2: On the left part, the graph of γ is inside $\mathrm{Int}(\Gamma^-)$ and γ is decreasing.

In the proof of Proposition 5.2, we will use the following identity, which comes from direct calculation and the fact that $LS = 0$:

$$\text{for all } (x,z) \in \mathbf{\Delta},\ \frac{\partial}{\partial x}\left(\frac{g_x(x,z)}{S'(x)}\right) = \frac{Lg(x,z)}{S'(x)}. \tag{5.14}$$

Proof. By definition of g and S, for any z, $f(0,z) = 0$. Then:

$$f_x(x,z) = g_x(x,z) - S(x)\frac{Lg(x,z)}{S'(x)} - g_x(x,z) = -S(x)\frac{Lg(x,z)}{S'(x)}.$$

Therefore, $f_x(x,\Gamma^0) < 0$ for any $x \in]0, \Gamma_\uparrow^{-1}(\Gamma^0)]$, thus $f(x,\Gamma^0) < 0$ if $x \in]0, \Gamma_\uparrow^{-1}(\Gamma^0)]$. On the other hand, if $z < \Gamma^0$, then $f(x,z) > 0$ for any $x \in (0, \Gamma^{-1}(z)]$, where $\Gamma^{-1}(z) > 0$.

By continuity of f, there exists $\varepsilon > 0$ and $x > 0$ such that for any $z \in]\Gamma^0 - \varepsilon, \Gamma^0]$, $f(x,z) < 0$. Therefore there exists $x \in]\Gamma_\downarrow^{-1}(z), \Gamma_\uparrow^{-1}(z)]$ satisfying $f(x,z) = 0$. Let z_0 be in such a neighborhood and let $x_0 \in]\Gamma_\downarrow^{-1}(z_0), \Gamma_\uparrow^{-1}(z_0)]$ satisfying $f(x_0, z_0) = 0$. By definition, $(x_0, z_0) \in \mathrm{Int}(\Gamma^+)$.

We consider now the following Cauchy problem:

$$\gamma_\downarrow'(x) = \frac{Lg(x,\gamma_\downarrow(x))S(x)}{S(x) - xS'(x) - \frac{(S(x))^2}{S(\gamma_\downarrow)}}, \tag{5.15}$$

with the additional condition $\gamma_\downarrow(x_0) = z_0$. ODE (5.15) is obtained by a formal derivation of the equation $f(x, \gamma(x)) = 0$. Indeed, assuming that γ is C^1, we see that:

$$f_x(x, \gamma(x)) + \gamma'(x) f_z(x, \gamma(x)) = 0$$

We compute:

$$\begin{aligned}
f_z(x, z) &= g_z(x, z) - g_{xz}(x, z)\frac{S(x)}{S'(x)} - z \\
&= z - x - \frac{S(x)}{S(z)}(z - x) + \frac{S(x)}{S'(x)}\left(1 + \frac{S'(x)(z - x)}{S(z)} - \frac{S(x)}{S(z)}\right) - z \\
&= -x + \frac{S(x)}{S'(x)} - \frac{(S(x))^2}{S'(x)S(z)}.
\end{aligned}$$

So we get:

$$\gamma'\left[-xS'(x) + S(x) - \frac{(S(x))^2}{S(\gamma)}\right] = S(x)Lg(x, \gamma).$$

As long as $x > 0$, $S(x) - xS'(x) - \frac{(S(x))^2}{S(\gamma_\downarrow)} \leq S(x) - xS'(x) < 0$, so the Cauchy problem is well defined (since $0 < x_0 \leq z_0$). The maximal solution will be defined on an interval (x_-, x_+), with $x_0 \in (x_-, x_+)$. We also have $\gamma'_\downarrow < 0$ as long as $(x, \gamma_\downarrow(x)) \in \mathrm{Int}(\Gamma^+)$ and $(x_0, z_0) \in \mathrm{Int}(\Gamma^+)$, so we have $graph(\gamma_\downarrow) \cap \Gamma \neq \emptyset$.

Since $\frac{\partial}{\partial z} Lg > 0$, $\frac{\partial^2}{\partial x^2} Lg < 0$ and $Lg(x, \Gamma(x)) = 0$ on $[0, \zeta]$, the implicit functions theorem implies that $\Gamma_\downarrow$ is C^1 with negative derivative. We also have if $(x_\Gamma, z_\Gamma) \in graph(\gamma_\downarrow) \cap \Gamma$, then $\gamma'_\downarrow(x_\Gamma) = 0$. Therefore (x_Γ, z_Γ) can only be on $graph(\Gamma_\uparrow)$. This implies that $x_- = 0$ and we can define $x^* = \inf\{x \geq x_0, (x, \gamma_\downarrow(x)) \in \Gamma\}$. $\gamma_\downarrow$ is defined on $(0, x^* + \varepsilon)$ for a certain $\varepsilon > 0$ and on $(0, x^*)$, $(x, \gamma_\downarrow(x)) \in \mathrm{Int}(\Gamma^+)$. Using (5.15), γ' is negative on $(0, x^*)$.

By construction $f(x, \gamma_\downarrow(x)) = \text{constant} = f(x_0, z_0) = 0$, $(x, \gamma_\downarrow(x)) \in \Gamma^+$ and $(x^*, \gamma_\downarrow(x^*)) \in graph(\Gamma_\uparrow)$.

Finally, since $\gamma_\downarrow$ is decreasing it has a limit at 0. The fact that $(x, \gamma_\downarrow(x)) \in \Gamma^+$ implies that $\gamma_\downarrow(0) \geq \Gamma^0$, and if we had $\gamma_\downarrow(0) > \Gamma^0$, then by continuity of $\gamma_\downarrow$, there would exist $x \in (0, \Gamma_\uparrow^{-1}(\Gamma^0)]$, such that $f(x, \Gamma^0) = 0$, which is impossible. So we have the result. $\square$

The function $\gamma_\downarrow$ defined in the previous proposition will be the second part of our boundary. We denote by $\gamma_\uparrow$ the boundary constructed in the previous paragraph. We now check that the two boundaries $\gamma_\uparrow$ and $\gamma_\downarrow$ do intersect. This is provided in the following proposition.

Proposition 5.3 *We have either $\gamma_\uparrow$ is increasing on $[0, +\infty)$, or $|graph(\gamma_\downarrow) \cap graph(\gamma_\uparrow)| = 1$. In the first case we write $\bar{x} = 0$ and $\bar{z} = \gamma_\uparrow(0)$. In the second case, we write $(\bar{x}, \bar{z}) = graph(\gamma_\downarrow) \cap graph(\gamma_\uparrow)$. In both cases we have $(\bar{x}, \bar{z}) \in \Gamma^+$ and $\{(x, \gamma_\uparrow(x));\ x > \bar{x} \text{ and } \gamma_\uparrow(x) > x\} \subset Int(\Gamma^+)$.*

Proof. $\gamma_\uparrow$ is increasing as long as $Lg(x, \gamma_\uparrow(x)) > 0$. By Proposition 5.1, if we do not have $\gamma_\uparrow$ increasing on $[0, +\infty)$, then there exists $x_0 \leq \zeta$ such that $Lg(x_0, \gamma_\uparrow(x_0)) = 0$ while $\gamma_\uparrow$ is increasing on $(x_0, +\infty)$. Since $\Gamma_\downarrow$ is decreasing on $(0, \zeta)$ while $\gamma_\uparrow$ is increasing as long as $(x, \gamma_\uparrow(x)) \in \mathrm{Int}(\Gamma^+)$, $(x, \gamma_\uparrow(x)) \in \Gamma^-$ on $(0, x_0)$.

On the other hand, $\gamma_\downarrow$ is defined on $[0, x^*]$, decreasing, continuous, and $(x, \gamma_\downarrow(x)) \in \mathrm{Int}(\Gamma^+)$ on $(0, x^*)$. Therefore we have $|graph(\gamma_\downarrow) \cap graph(\gamma_\uparrow)| = 1$, this intersection is in Γ^+ and by construction the last property is immediate.

If $\gamma_\uparrow$ is increasing on $[0, +\infty)$, then $(x, \gamma_\uparrow(x)) \in \Gamma^+$ for all $x > 0$, so by continuity of $\gamma_\uparrow$ and since Γ^+ is a closed set, it is still true for $x = 0$. $\square$

From now on, we write γ the concatenation of $\gamma_\downarrow$ and $\gamma_\uparrow$, which is continuous and piecewise C^1:

$$\gamma(x) = \begin{cases} \gamma_\downarrow(x) \text{ if } x < \bar{x} \\ \gamma_\uparrow(x) \text{ if } x \geq \bar{x} \end{cases}$$

We also introduce:

$$\phi_\downarrow = \gamma_\downarrow^{-1} \quad \text{and} \quad \phi_\uparrow = \gamma_\uparrow^{-1}. \tag{5.16}$$

Notice that Proposition 5.1 (respectively Proposition 5.2) implies that $\phi_\uparrow$ (resp. $\phi_\downarrow$) is C^1 on $\{z > \bar{z},\ \phi_\uparrow(z) < z\}$ (resp. on $(\bar{z}, \Gamma^0)$), with positive (resp. negative) derivative.

Notice that if $\gamma_\downarrow$ is degenerate, then $\gamma = \gamma_\uparrow$.

6 Definition of v and verification result

Now we are able to define our candidate function v and we will prove that it is the value function V defined by (2.7).

Let us first decompose $\mathbf{\Delta}$ into four different sets. We define:

$$\begin{aligned} A_1 &= \{(x, z),\ x \in [0, \bar{x}[\text{ and } \bar{z} < z < \gamma(x)\} \\ A_2 &= \{(x, z),\ x \geq \bar{x} \text{ and } \bar{z} < z < \gamma(x)\} \\ A_3 &= \{(x, z),\ 0 \leq x \leq z \leq \bar{z}\} \\ A_4 &= \{(x, z),\ x \geq 0 \text{ and } z \geq \gamma(x)\}. \end{aligned}$$

(A_1, A_2, A_3, A_4) is a partition of $\mathbf{\Delta}$. Notice that if $(x, z) \in A_2$, then by Proposition 5.1-(iii), $x \leq \Gamma^\infty$, and recall that $\bar{x} < \bar{z}$ were defined in Proposition 5.3, while $\phi_\downarrow$ and $\phi_\uparrow$ were defined by (5.16). Notice also that A_2 is not necessarily connected.

We refer to Figure 3 for a better understanding of the different areas. Let

$$K := \int_{\bar{z}}^{\infty} \frac{u}{S(u)} du - \frac{g_x(\bar{x}, \bar{z})}{S'(\bar{x})}, \tag{6.1}$$

we define v in the following way:

$$v(x,z) = \frac{z^2}{2} + g_x(\phi_\downarrow(z),z)\frac{S(x)}{S'(\phi_\downarrow(z))} \text{ if } (x,z) \in A_1, \tag{6.2}$$

$$v(x,z) = g(\phi_\uparrow(z),z) + g_x(\phi_\uparrow(z),z)\frac{S(x) - S(\phi_\uparrow(z))}{S'(\phi_\uparrow(z))} \text{ if } (x,z) \in A_2, \tag{6.3}$$

$$v(x,z) = \frac{z^2}{2} + S(x)\left[\int_z^\infty \frac{u}{S(u)}du - K\right] \text{ if } (x,z) \in A_3, \tag{6.4}$$

$$v(x,z) = g(x,z) \text{ if } (x,z) \in A_4. \tag{6.5}$$

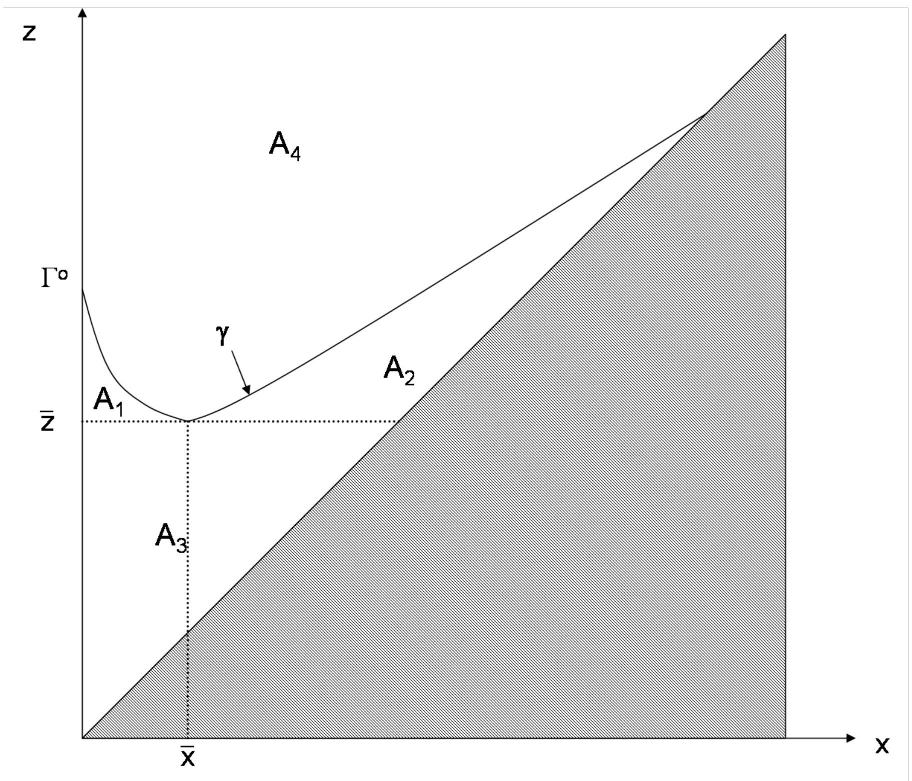


Figure 3: The different areas

Theorem 6.1 *Let the coefficient α satisfy Conditions* (2.2) *and* (4.1)*. Let γ be given by Proposition 5.1 and v be defined by* (6.2) *to* (6.5)*. Then $v = V$ and $\theta^* = \inf\{t \geq 0; Z_t \geq \gamma(X_t)\}$ is an optimal stopping time.*
Moreover if τ is another optimal stopping time, then $\theta^* \leq \tau$ a.s.

Proof. From Proposition 5.1, Lemmas 6.1 and 6.2 below and Propositions 6.1 and 6.2 below, v and γ satisfy the assumptions of Theorem 3.1. $\square$

First we need to prove that v is sufficiently regular, in order to be able to apply Itô's formula. We denote by $\bar{A}$ the closure of a set A.

Lemma 6.1 *v is C^0 w.r.t (x,z), C^1 w.r.t x and piecewise $C^{2,1}$ w.r.t. (x,z). More precisely, except on $\cup_{i\neq j}(\bar{A}_i \cap \bar{A}_j)$, it is $C^{2,1}$.*

Proof. From the definition of v, $\phi_\downarrow$ and $\phi_\uparrow$, it is immediate that v can be extended as a $C^{2,1}$ function on any $\bar{A}_i$.

Let us write v_i the expression of v on $\bar{A}_i$. Since $\phi_\downarrow$ satisfies (5.13), it is immediate to see that v is C^0 w.r.t. (x,z) and C^1 w.r.t. x on the boundary (v_1 with v_4 and v_2 with v_4). On $z=\bar{z}$, it is easy to check that the expressions of v_2 and v_3 coincide. It is also true for v_1 and v_3 since $\phi_\downarrow$ satisfies (5.13) and $\bar{x}=\phi_\downarrow(\bar{z})$. It is straightforward that it is also C^1 and even C^2 w.r.t x. □

We now show that v satisfies the limit conditions.

Lemma 6.2 *$\forall z \geq 0,\ v(0,z)=\frac{z^2}{2}$ and $v_z(z,z)=0$.*

Proof. Since $S(0)=0$, $v(0,z)=\frac{z^2}{2}$ is immediate.

For $(z,z)\in \mathrm{Int}(A_4)$, since $g_z(z,z)=0$, we have $v_z(z,z)=0$. For $(z,z)\in\mathrm{Int}(A_3)$ it is immediate that $v_z(z,z)=0$. For $(z,z)\in \mathrm{Int}(A_2)$, since γ_2 satisfies ODE (5.1), $\phi_\uparrow'(z)Lg(\phi_\uparrow(z),z)=1-\frac{S(\phi_\uparrow(z))}{S(z)}$. We then compute:

$$\begin{aligned} v_z(z,z) &= g_z(\phi_\uparrow(z),z)+g_{xz}\frac{S(z)-S(\phi_\uparrow(z))}{S'(\phi_\uparrow(z))}+\phi_\uparrow'(z)Lg(\phi_\uparrow(z),z)\frac{S(z)-S(\phi_\uparrow(z))}{S'(\phi_\uparrow(z))} \\ &= -\left(1-\frac{S(\phi_\uparrow(z))}{S(z)}\right)\frac{S(z)-S(\phi_\uparrow(z))}{S'(\phi_\uparrow(z))}+\left(1-\frac{S(\phi_\uparrow(z))}{S(z)}\right)\frac{S(z)-S(\phi_\uparrow(z))}{S'(\phi_\uparrow(z))}=0. \end{aligned}$$

To complete the proof, we need to show that $v_z(\bar{z},\bar{z})=0$ and $v_z(\Gamma^\infty,\Gamma^\infty)=0$ if $\Gamma^\infty<\infty$. The previous computations and the definition of v on A_3 and A_4 show that at those points, $v_z(z,z)$ has right and left limits that are both equal to 0, so we have the result. □

Proposition 6.1 *Let the coefficient α satisfy Conditions (2.2) and (4.1). Then the function v is bounded from below and $\lim_{z\to\infty} v(z,z)-g(z,z)=0$.*

Proof. If $\Gamma^\infty<\infty$, it is immediate since in this case, by Proposition 5.1-(iii), $v=g$ outside a compact set, v is continuous and g is non-negative. So let us focus on the case $\Gamma^\infty=\infty$. If (4.1) is satisfied, by Proposition 4.3, we know that α is bounded. We write $\alpha\leq M$.

We first prove that v is bounded from below and that $v(z,z)-g(\phi_\uparrow(z),z)\to 0$ as $z\to\infty$. A_1 is bounded because of the definition of $\gamma_\downarrow$, and A_3 is bounded by definition. Since $v=g$ on A_4 and $g\geq 0$, we only need to check that v is bounded from below on A_2.

On the set $\{(x,\gamma_\uparrow(x));\ x\in[\bar{x},\infty)\}$, $v=g$, and for $(x,z)\in A_2$, we have:

$$v(x,z)=g(\phi_\uparrow(z),z)+g_x(\phi_\uparrow(z),z)\frac{S(x)-S(\phi_\uparrow(z))}{S'(\phi_\uparrow(z))}. \tag{6.6}$$

In particular, we see that for each z, $v(.,z)$ is monotonic on $[\phi_\uparrow(z),z]$. Therefore, since $v(\phi_\uparrow(z),z)\geq g(\phi_\uparrow(z),z)\geq 0$, it is sufficient to check that v is bounded from below on the diagonal $\{(z,z);\ z\in[\bar{x},\infty)\}$.

We compute:

$$g_x(\phi_\uparrow(z), z) = -(z - \phi_\uparrow(z)) + S'(\phi_\uparrow(z)) \int_z^\infty \frac{u - \phi_\uparrow(z)}{S(u)} du - S(\phi_\uparrow(z)) \int_z^\infty \frac{du}{S(u)}.$$

From Proposition 5.1 we know that $\lim_{x\to\infty} \gamma_\uparrow(x) - x = 0$, so that $\lim_{z\to\infty} z - \phi_\uparrow(z) = 0$. Using Proposition 4.1, the fact that $\phi_\uparrow(z) < z$ since $\Gamma^\infty = \infty$, and the increase of S', we have as $z \to \infty$:

$$S'(\phi_\uparrow(z)) \int_z^\infty \frac{u - \phi_\uparrow(z)}{S(u)} du \sim S'(\phi_\uparrow(z)) \frac{z - \phi_\uparrow(z)}{S'(z)} = O(1),$$
$$S(\phi_\uparrow(z)) \int_z^\infty \frac{du}{S(u)} \sim \frac{S'(\phi_\uparrow(z))}{\alpha(\phi_\uparrow(z)) S'(z)} = O(1),$$

so that $g_x(\phi_\uparrow(z), z) = O(1)$.

Since $\alpha \leq M$ and S' is increasing:

$$S'(z) = S'(\phi_\uparrow(z)) e^{\int_{\phi_\uparrow(z)}^z \alpha(u) du} \leq S'(\phi_\uparrow(z)) e^{M(z - \phi_\uparrow(z))},$$

so that:

$$\begin{aligned} S(z) - S(\phi_\uparrow(z)) &\leq (z - \phi_\uparrow(z)) S'(z) \\ &\leq (z - \phi_\uparrow(z)) S'(\phi_\uparrow(z)) e^{M(z - \phi_\uparrow(z))}, \end{aligned} \tag{6.7}$$

and therefore: $0 \leq \frac{S(z) - S(\phi_\uparrow(z))}{S'(\phi_\uparrow(z))} \leq (z - \phi_\uparrow(z)) e^{M(z - \phi_\uparrow(z))} = \circ(1)$.

Since v is continuous and $g \geq 0$, by (6.6) we see that v is bounded from below and $v(z, z) - g(\phi_\uparrow(z), z) \to 0$.

Finally, we show that $g(z, z) - g(\phi_\uparrow(z), z) \to 0$. Indeed we compute:

$$\begin{aligned} g(z, z) - g(\phi_\uparrow(z), z) = -\frac{(z - \phi_\uparrow(z))^2}{2} &+ \Big(S(z) - S(\phi_\uparrow(z))\Big) \int_z^\infty \frac{u - z}{S(u)} du \\ &- S(\phi_\uparrow(z)) \int_z^\infty \frac{z - \phi_\uparrow(z)}{S(u)} du. \end{aligned}$$

Using Proposition 4.1-(ii) and (6.7), we get:

$$\begin{aligned} \Big(S(z) - S(\phi_\uparrow(z))\Big) \int_z^\infty \frac{u - z}{S(u)} du &\sim \frac{S(z) - S(\phi_\uparrow(z))}{\alpha(z) S'(z)} \\ &\leq \frac{z - \phi_\uparrow(z)}{\alpha(z)} = \circ(1). \end{aligned}$$

Using again Proposition 4.1, we also get:

$$S(\phi_\uparrow(z)) \int_z^\infty \frac{z - \phi_\uparrow(z)}{S(u)} du \sim (z - \phi_\uparrow(z)) \frac{S'(\phi_\uparrow(z))}{\alpha(\phi_\uparrow(z)) S'(z)} = \circ(1),$$

and as a consequence:

$$g(z, z) - g(\phi_\uparrow(z), z) = \circ(1).$$

Therefore we finally have $\lim_{z\to\infty} v(z, z) - g(z, z) = 0$. □

The final property of v required by the verification Theorem 3.1 is the following.

Proposition 6.2 *Let the coefficient α satisfy Conditions* (2.2) *and* (4.1). *Then $v \leq g$ on $\boldsymbol{\Delta}$ and $v < g$ on the continuation region $\{(x,z) \in \boldsymbol{\Delta};\ x > 0$ and $z < \gamma(x)\}$.*

Proof.

On A_1:

For $\bar{z} \leq z < \Gamma^0$ and $0 \leq x < \phi_\downarrow(z)$, we have:

$$\begin{aligned} v(x,z) - g(x,z) &= \frac{z^2}{2} + g_x(\phi_\downarrow(z), z)\frac{S(x)}{S'(\phi_\downarrow(z))} - g(x,z), \\ v_x(x,z) - g_x(x,z) &= g_x(\phi_\downarrow(z), z)\frac{S'(x)}{S'(\phi_\downarrow(z))} - g_x(x,z) \\ &= S'(x)\int_x^{\phi_\downarrow(z)} \frac{Lg(u,z)}{S'(u)}du, \end{aligned}$$

where we used (5.14) for the last equality.

For $\bar{z} \leq z < \Gamma^0$, $(0,z) \in \Gamma^-$ while $(\phi_\downarrow(z), z) \in \Gamma^+$, so we can a priori have three behaviors for $v(.,z) - g(.,z)$:
- it is increasing on $[0, \phi_\downarrow(z)]$,
- or it is decreasing on $[0, \phi_\downarrow(z)]$,
- or it is decreasing on $[0, \delta)$ and increasing on $(\delta, \phi_\downarrow(z)]$, for a certain $\delta \in (0, \phi_\downarrow(z))$.

Since $v(0,z) = g(0,z)$ and $v(\phi_\downarrow(z), z) = g(\phi_\downarrow(z), z)$, only the last behavior can occur and $v \leq g$ on A_1. Moreover $v < g$, except if $x = 0$ or $x = \phi_\downarrow(z)$.

On A_2:

For $x > \phi_\uparrow(z)$ and $\bar{z} \leq z < \Gamma^\infty$, we compute:

$$v(x,z) - g(x,z) = g(\phi_\uparrow(z), z) + g_x(\phi_\uparrow(z), z)\frac{S(x) - S(\phi_\uparrow(z))}{S'(\phi_\uparrow(z))} - g(x,z).$$

So, similarly:

$$v_x(x,z) - g_x(x,z) = -S'(x)\int_{\phi_\uparrow(z)}^{x} \frac{Lg(u,z)}{S'(u)}du$$

Here again only three behaviors are a priori possible, for $(v-g)(.,z)$: increasing on $[\phi_\uparrow(z), z]$, decreasing on $[\phi_\uparrow(z), z]$ or decreasing on $[\phi_\uparrow(z), \delta)$ and increasing on $(\zeta, z]$ for a certain $\delta \in (\phi_\downarrow(z), z)$. As $v(\phi_\uparrow(z), z) = g(\phi_\uparrow(z), z)$, we need only to consider $v(z,z) - g(z,z)$.

We write $n(z) = v(z,z) - g(z,z)$. Since $v_z(z,z) = g_z(z,z) = 0$:

$$\begin{aligned} \frac{\partial}{\partial z}(v(z,z) - g(z,z)) = n'(z) &= v_x(z,z) - g_x(z,z) \\ &= -S'(z)\int_{\phi_\uparrow(z)}^{z} \frac{Lg(u,z)}{S'(u)}du. \end{aligned}$$

We find the same expression as before, with $x = z$. If $n'(z) < 0$, we have $\int_{\phi_\uparrow(z)}^{z} \frac{Lg(u,z)}{S'(u)}du > 0$ which implies that for any $x \in (\phi_\uparrow(z), z]$, $\int_{\phi_\uparrow(z)}^{x} \frac{Lg(u,z)}{S'(u)}du > 0$. Therefore $(v-g)(.,z)$ is decreasing on $[\phi_\uparrow(z), z]$, and as $(v-g)(\phi_\uparrow(z), z) = 0$, we get $n(z) < 0$ if $\phi_\uparrow(z) < z$.

Assume that there exists $z \in [\bar{z}, \Gamma^\infty)$ such that $n(z) \geq 0$ and $\phi_\uparrow(z) < z$. Then, from the previous argument, $n'(z) \geq 0$. Since n is continuous this implies that n is nondecreasing on any connected subset of $\{z' \geq z,\ \gamma(z') > z'\}$. Let $a := \inf\{z' > z;\ \gamma(z') = z'\}$. $a < \infty$ is impossible since $v(a,a) = g(a,a)$, and if $a = \infty$, Proposition 6.1 gives $\lim_{z\to\infty} n(z) = 0$, so again this is impossible. Finally, $n(z) < 0$ if $\phi_\uparrow(z) < z$. Therefore $v \leq g$ on A_2 and $v < g$ except if $x = \phi_\uparrow(z)$.

On A_3:
Recall the definition of K given by (6.1). For $x \leq z < \bar{z}$, we have:

$$v(x,z) - g(x,z) = \frac{z^2}{2} - KS(x) + \frac{(z-x)^2}{2} + xS(x)\int_z^{+\infty} \frac{du}{S(u)},$$

$$\text{so } v_z(x,z) - g_z(x,z) = x\left(1 - \frac{S(x)}{S(z)}\right).$$

The latter expression is non-negative and positive if $x \neq 0$. Since v and g are continuous, the result for A_1 and A_2 tells us that $v(.,\bar{z}) \leq g(.,\bar{z})$, so that $v \leq g$ on A_3 and $v < g$ if $x \neq 0$. □

In the next section, we will provide a few examples.

7 Examples

7.1 Brownian motion

In this case, $\alpha(x) = 0$ and $S(x) = x$. As (2.14) will never be satisfied for a non-decreasing and convex function ℓ, proposition (2.1) tells us that V and g will be infinite if ℓ satisfies (2.16). But moreover we have the following result.

Proposition 7.1 *For any $0 < x \leq z$ and any convex and nondecreasing function ℓ, we have:*
(i) $\mathbb{E}_{x,z} T_0 = +\infty$,
(ii) $\mathbb{E}_{x,z} Z_{T_0} = \mathbb{E}_{x,z}(Z_{T_0})^2 = +\infty$,
(iii) *V and g are infinite everywhere except for $x = 0$.*

Proof. (i) This is a very well-known result, but we give a proof for completeness reasons. By direct calculation:

$$\mathbb{P}_x[T_0 \geq t] = \mathbb{P}_x\left[\inf_{[0,t]} X_u > 0\right] = \mathbb{P}_0[X_t^* > x] = 2\mathbb{P}_0[X_t > x].$$

So that $\mathbb{E}_{x,z} T_0 = \int_0^\infty \frac{xte^{-\frac{x^2}{2t\sigma^2}}dt}{\sqrt{2\pi}t^{\frac{3}{2}}\sigma} = +\infty$ as $\frac{2e^{-\frac{x^2}{2t\sigma^2}}}{\sqrt{2\pi t}\sigma} \sim \sqrt{\frac{2}{\pi t\sigma^2}}$ when $t \to +\infty$.
(ii) It is sufficient to prove that $\mathbb{E}_{x,z} Z_{T_0} = \infty$. By direct calculation, using (2.11) provides:

$$\begin{aligned}\mathbb{E}_{x,z} Z_{T_0} &= \left(1 - \frac{S(x)}{S(z)}\right) z + \int_z^\infty y \frac{S(x)S'(y)}{S^2(y)} dy \\ &= (z-x) + \int_z^\infty \frac{x}{y} dy = +\infty.\end{aligned}$$

(iii) Since ℓ is non-decreasing and convex, there exist $C > 0$ and $D \in \mathbb{R}$, such that $\ell(x) \geq Cx + D$. Therefore (ii) implies that g is infinite everywhere except for $x = 0$. Now as in the proof of Proposition 2.1, this implies that V is infinite everywhere (except if $x = 0$). □

7.2 Brownian motion with negative drift

Now we consider the following diffusion, for constant $\mu < 0$ and $\sigma > 0$:

$$dX_t = \mu dt + \sigma dW_t.$$

Therefore $\alpha(x) = -\frac{2\mu}{\sigma^2} = \alpha > 0$, $S(x) = \frac{e^{\alpha x}-1}{\alpha}$, and $S'(x) = e^{\alpha x}$.

We have an interesting homogeneity result for this process, as long as ℓ is a power function, which allows us to assume that $\alpha = 1$. In the following statement, we denote by γ_α the corresponding boundary.

Proposition 7.2 *Let $\alpha > 0$ and $p > 1$ be given, and consider the following loss function $\ell(x) = x^p$. Then:*

$$\gamma_\alpha(z) = \frac{\gamma_1(\alpha z)}{\alpha}.$$

Proof. Let X be a drifted Brownian motion with parameter $\alpha_X = \alpha$, and define $\bar{X} = \alpha X$. The dynamics of $\bar{X}$ is

$$d\bar{X}_t = \alpha dX_t = \alpha\mu dt + \alpha\sigma dW_t.$$

So that $\alpha_{\bar{X}} = \frac{-2\mu\alpha}{\sigma^2\alpha^2} = 1$. Let $\bar{Z}$ be the corresponding running maximum, started from αz. Then $\bar{Z} = \alpha Z$, $T_0(X) = T_0(\bar{X}) = T_0$ and for any θ:

$$\mathbb{E}_{kx,kz}(\bar{Z}_{T_0} - \bar{X}_\theta)^p = \alpha^p \mathbb{E}_{x,z}(Z_{T_0} - X_\theta)^p.$$

This equality implies that if τ is optimal for one problem, it is also optimal for the other one. Together with the minimality of θ^*, it means that:

$$Z_t = \gamma_\alpha(X_t) \Leftrightarrow \bar{Z}_t = \gamma_1(\bar{X}_t) \Leftrightarrow \alpha Z_t = \gamma_1(kX_t),$$

which completes the proof. □

In the quadratic case $\ell(x) = \frac{x^2}{2}$, we have $Lg(x,z) = 1 + \alpha(z-x) + (1+e^{\alpha x})\ln(1-e^{-\alpha z})$.

We can see that $\frac{\partial}{\partial x}Lg < 0$, so that Γ is increasing (ie $\zeta = 0$). Moreover, for any $x > 0$, $\ln(1-x) < -x$, so that for $z > 0$, $Lg(z,z) < -e^{-\alpha z} < 0$, so that $\Gamma^\infty = +\infty$.

Figure 4 below is a numerical computation of γ for $\ell(x) = \frac{x^2}{2}$. As Γ is increasing, γ is necessarily increasing too ($\gamma_\downarrow$ is degenerate). Even though it does not affect the shape because of Proposition 7.2, this plot was computed for $\alpha = 1$.

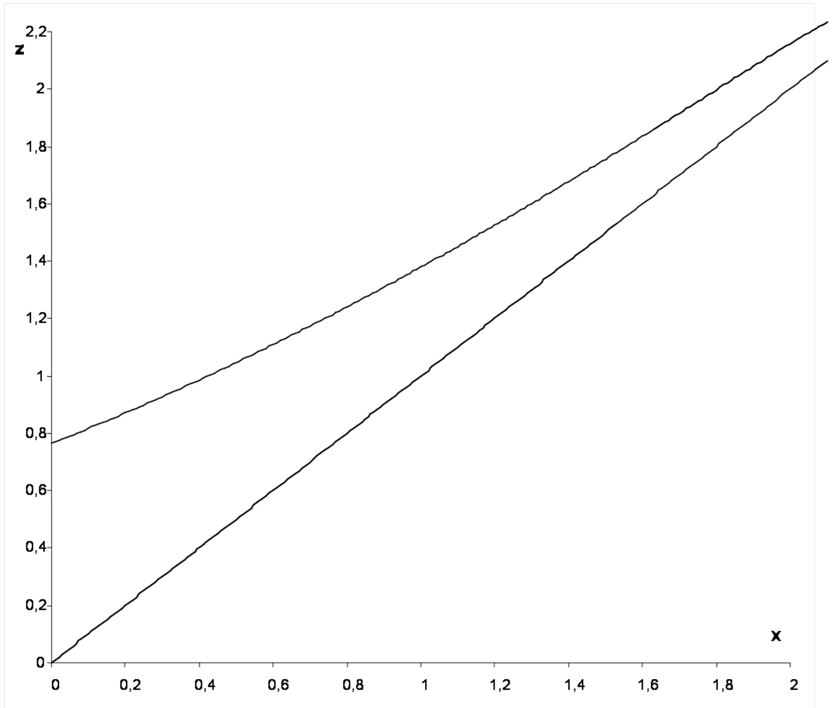


Figure 4: γ for a Brownian motion with negative drift and $\ell(x) = \frac{x^2}{2}$

7.3 The CIR-Feller process

Let $b \geq 0$, $\mu < 0$ and $\sigma > 0$, then the dynamics of X is:

$$dX_t = \mu X_t dt + \sigma\sqrt{b + X_t} dW_t.$$

Here, $\alpha(x) = \alpha\frac{x}{x+b}$ with $\alpha > 0$. In the degenerate case $b = 0$, we are reduced to the context of the Brownian motion with negative drift. We then focus on the case $b > 0$ with a quadratic loss function $\ell(x) = \frac{x^2}{2}$. Proceeding as in the proof of Proposition 4.3, we can see that $\Gamma^\infty < \infty$, unlike in the case $b = 0$.

Moreover, as $x \to 0$, $\alpha(x) \sim \frac{\alpha x}{b}$, $\alpha'(x) \sim \frac{\alpha}{b}$, so that we can see that for any $z > 0$, $\frac{\partial}{\partial x} Lg > 0$ for x small enough, which means that $\Gamma_\downarrow$ is not degenerate, or equivalently that $\zeta > 0$.

7.4 Ornstein-Uhlenbeck process

The dynamics of X is now given by:

$$dX_t = \mu X_t dt + \sigma dW_t,$$

so that $\alpha(x) = \alpha x$, $S'(x) = e^{\alpha\frac{x^2}{2}}$.

This case and the Brownian motion with negative drift case can be seen as the extreme cases of our framework. Indeed here $\alpha(x) = \alpha x$ is the "most increasing" concave function, while $\alpha(x) = \alpha$ is the "least non-decreasing" function.

As for the Brownian motion with negative drift, we have an homogeneity result for this process, as long as ℓ is a power function, which allows us to assume that $\alpha(x) = x$.

Proposition 7.3 *Let $\alpha(x) = \alpha x$ with $\alpha > 0$ and $\ell(x) = x^p$ with $p > 1$. Then the corresponding boundary γ_α satisfies:*

$$\gamma_\alpha(z) = \frac{\gamma_1(z\sqrt{\alpha})}{\sqrt{\alpha}}$$

Proof. We follow the proof in the case of a Brownian motion with negative drift. Let X be process with $\alpha_X(x) = \alpha x$. Then the process $\bar{X} = \sqrt{\alpha}X$ is such that $\alpha_{\bar{X}} = 1$. Denote by $\bar{Z}$ the corresponding running maximum process. Then $\bar{Z} = \sqrt{\alpha}Z$, $T_0(X) = T_0(\bar{X}) = T_0$ and for any θ:

$$\mathbb{E}_{\sqrt{\alpha}x,\sqrt{\alpha}z}(\bar{Z}_{T_0} - \bar{X}_\theta)^p = \alpha^{\frac{p}{2}}\mathbb{E}_{x,z}(Z_{T_0} - X_\theta)^p$$

Then by the minimality of θ^* we have:

$$X_t = \gamma_\alpha(Z_t) \Leftrightarrow \bar{X}_t = \gamma_1(\bar{Z}_t) \Leftrightarrow \sqrt{\alpha}X_t = \gamma_1(kZ_t),$$

which provides the required result. □

Then, focusing on the case $\ell(x) = \frac{x^2}{2}$, we show that Γ is decreasing in a neighborhood of 0, so that $\zeta > 0$ and that $\Gamma^\infty < +\infty$.

Proposition 7.4 *For an Ornstein-Uhlenbeck process:*
- *$Lg(x, \Gamma^0) > 0$ for $x > 0$ in a neighborhood of 0, therefore $\Gamma_\downarrow$ is not degenerate,*
- *$Lg(z, z) > 0$ in a neighborhood of $+\infty$, therefore $\Gamma^\infty < +\infty$.*

Proof. Since $\alpha(x) \to \infty$ as $x \to \infty$, Proposition 4.3 implies that $\Gamma^\infty < \infty$.

If x is small, we have the $S(x) \sim x$, $S'(x) = 1 + S''(0)x + \circ(x) = 1 + \circ(x)$ and by definition of Γ^0, $\int_{\Gamma^0}^\infty \frac{du}{S(u)} = \frac{1}{2}$. Therefore, as $x \to 0$, we can write:

$$Lg(x, \Gamma^0) = 1 + \alpha x\Gamma^0 - 1 + \circ(x).$$

Since $\alpha > 0$ and $\Gamma^0 > 0$ by Proposition 4.2, $Lg(x, \Gamma^0) > 0$ for $x > 0$ and sufficiently small. □

Finally, Figure 5 below is a numerical computation of the boundary γ for $\ell(x) = \frac{x^2}{2}$. While we do not prove it, we can see that γ is in this case decreasing first and then increasing. Although it does not affect the shape because of Proposition 7.3, it was computed for $\alpha = 1$.

8 Extension to general loss functions

Except for sections 2 and 3, the previous analysis only considered the case of the quadratic loss function $\ell(x) = \frac{x^2}{2}$. In fact, as the reader has probably noticed, the quadratic loss function plays a special role here, since we then have $\ell^{(3)}(x) = 0$, which simplifies a lot the study of the set Γ^+, as well as the asymptotic behavior of Lg. Some crucial properties that

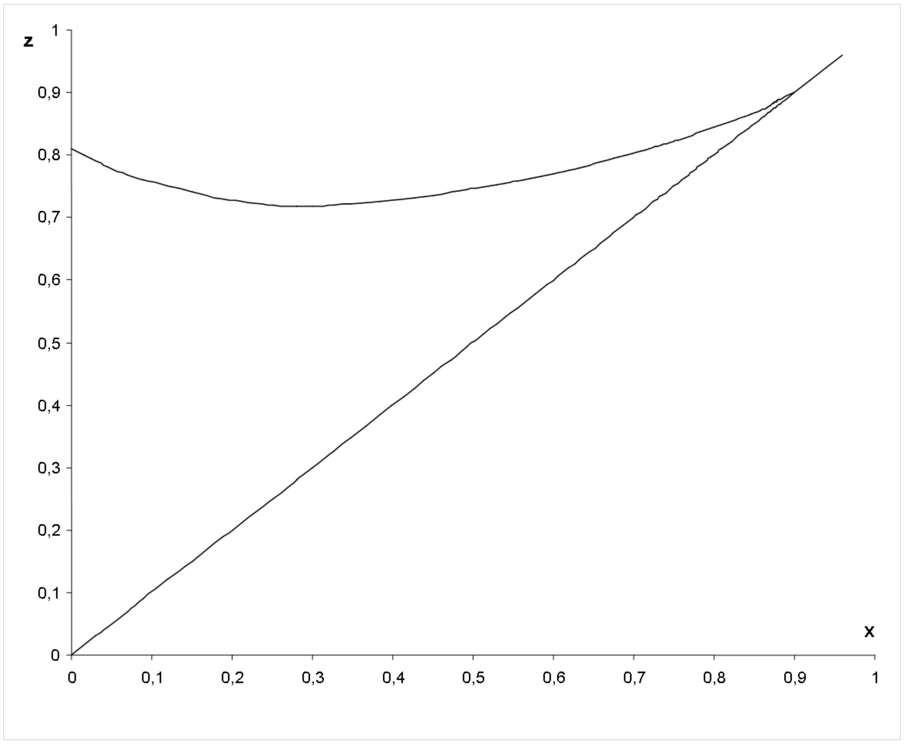


Figure 5: γ for an OU process with $\ell(x) = \frac{x^2}{2}$

we estabished in the quadratic case seem very hard to derive in the general case stated in the sections 2 and 3.

Nevertheless, under additional assumptions, our results still hold true in a more general framework. We explain here how to do so.

Let us compute:

$$\begin{aligned} Lg(x,z) =& \ell''(z-x) + \alpha(x)\ell'(z-x) - (2S'(x) - \alpha(x)S(x)) \int_z^\infty \frac{\ell''(u-x)}{S(u)} du \\ &+ S(x) \int_z^\infty \frac{\ell^{(3)}(u-x)}{S(u)} du \end{aligned}$$

Since $\ell''(x) > 0$ for $x > 0$ and $\ell^{(3)} \geq 0$, for any $x \geq 0$, $z \mapsto \frac{\partial}{\partial z} Lg(x,z)$ is increasing. Moreover, we have $\ell'(x) \to \infty$ as $x \to \infty$, so that for any $x \geq 0$, $\lim_{z\to\infty} Lg(x,z) > 0$. As a consequence, $\Gamma^+ \neq \emptyset$, the definition of Γ in (4.3) can be extended.

The main problem is that Lg is no longer concave with respect to x, and it is not clear how to show that Γ is U-shaped. In fact Proposition 4.2 (i) and Proposition 4.3 are crucial but we are unable to prove them in general. Therefore we make the following additional assumptions:

$$\exists \zeta \geq 0, \text{ such that } \Gamma \text{ is decreasing on } [0,\zeta] \text{ and increasing on } [\zeta,+\infty) \tag{8.1}$$

$$\text{if } \lim_{x\to\infty} \alpha(x) = \infty, \text{ then } \Gamma^\infty < \infty. \tag{8.2}$$

$\Gamma^0 < \Gamma^\infty$ from Proposition 4.2-(iii) is not true in general, but this is not important. It just means that we have a new possibility for the shape of γ: $\gamma_\uparrow(x) = x$ for every $x \geq \bar{x}$, as we will explain later.

Then ODE (5.1) is replaced by:

$$\gamma' = \frac{Lg(x,\gamma)}{\ell''(\gamma - x)\left(1 - \frac{S(x)}{S(\gamma)}\right)}. \tag{8.3}$$

Since $\ell'' > 0$, the Cauchy problem is well defined for any $x_0 > 0$ and $\gamma(x_0) > x_0$, and the maximal solution is defined as long as $\gamma(x) > x$.

In order to prove Proposition 5.1, we need asymptotic results as in Proposition 4.1:

If ℓ is not the quadratic loss function, we will make the following assumptions:

$$\ell \text{ is } C^3,\ \ell' > 0,\ \ell'' > 0,\ \ell^{(3)} \geq 0 \text{ and } \ell,\ \ell',\ \ell'' \text{ satisfy (2.14)} \tag{8.4}$$

$$K_1 := \sup_{y \geq 0} \frac{\ell^{(3)}(y)}{\ell''(y)} < \infty \text{ and } \lim_{x\to\infty} \alpha(x) > K_1 \tag{8.5}$$

$$K_2 := \sup_{y \geq 0} \frac{\ell''(y)}{\ell'(y)} < \infty \text{ and } \lim_{x\to\infty} \alpha(x) > K_2. \tag{8.6}$$

Notice that those assumptions are satisfied for exponential loss functions $\ell(x) = \lambda e^x$ with $\lambda > 0$ or for power loss functions of the form $\lambda(x+\varepsilon)^p$ with $\varepsilon > 0$ and $p \geq 2$.

Proposition 8.1 *Assume* (8.4)-(8.6). *Let φ be a measurable function such that* $0 \leq \varphi(z) \leq z$ *for all z (large enough). Then we have the following asymptotic behaviors, as $z \to \infty$:*
(i) *there exists a bounded function δ (depending on φ) satisfying $\delta(z) \geq 1$, for z large enough, and such that:*

$$\int_z^\infty \frac{\ell''(u - \varphi(z))}{S(u)} du \sim \delta(z) \frac{\ell''(z - \varphi(z))}{S'(z)};$$

(ii) *there exists a bounded function ν satisfying $\nu(z) \geq 1$, for z large enough, and such that:*

$$\int_z^\infty \frac{\ell'(u - \varphi(z))}{S(u)} du \sim \nu(z) \frac{\ell'(z - \varphi(z))}{S'(z)}.$$

Moreover if $\lim_{x\to\infty} \alpha(x) = \infty$, then for any function φ, δ and ν are constant and equal to 1.

Proof. The proof is given in appendix. □

If $\varepsilon > 0$, from Proposition 8.1 (i), there is a bounded function δ such that, as $x \to \infty$, we get:

$$Lg(x,(1+\varepsilon)x) \geq \ell''(\varepsilon x)\left[1 - \delta((1+\varepsilon)x)\frac{S'(x)}{S'((1+\varepsilon)x)}\right] + \alpha(x)\ell'(\varepsilon x) + \circ(1).$$

Now $\frac{S'(x)}{S'((1+\varepsilon)x)} \to 0$ as $x \to \infty$, while $\alpha(x)\ell'(\varepsilon x) \to \infty$, so that $Lg(x,(1+\varepsilon)x) \geq 1+3\varepsilon$ for x large enough, and the proof of Lemma 5.1-(iii) can be achieved as in the quadratic case.

The other statements of Lemma 5.1 as well as Lemma 5.2 can be proved using the same arguments as in the quadratic case. Finally we make a last assumption:

$$\text{either } \alpha(x) \to \infty \text{ as } x \to \infty, \tag{8.7}$$
$$\text{or in Proposition 8.1 (ii), for any } a > 0, \text{ and } \varphi(z) = z - a, \ \delta \equiv 1. \tag{8.8}$$

Thanks to this, the proof of Proposition 5.1 is still valid. Indeed for $a > 0$, we compute:

$$\begin{aligned} Lg(x,x+a) =& \ell''(a) + \alpha(x)\ell'(a) - S'(x)\int_{x+a}^{\infty} \frac{\ell''(u-x)}{S(u)}du \\ &+ S(x)\int_{x+a}^{\infty} \frac{\ell^{(3)}(u-x)}{S(u)}du + \circ(1) \\ &\geq \ell''(a) + \alpha(x)\ell'(a) - \eta\ell''(a)\frac{S'(x)}{S'(x-a)} + \circ(1). \end{aligned}$$

If $\alpha(x) \to \infty$ as $x \to \infty$, then the previous expression explodes to infinity as well. If $\alpha \to M$ with $M > 0$, then our assumption guarantees that $\eta = 1$, so that:

$$\begin{aligned} \frac{Lg(x,x+a)}{\ell''(a)\left(1 - \frac{S(x)}{S(x+a)}\right)} &= \frac{\ell''(a)(1-e^{-aM}) + M\ell'(a)}{\ell''(a)(1-e^{-aM})} + \circ(1) \\ &= 1 + \frac{M\ell'(a)}{\ell''(a)(1-e^{-aM})} + \circ(1). \end{aligned}$$

Since $\frac{M\ell'(a)}{\ell''(a)(1-e^{-aM})} > 0$, as in the proof of Proposition 5.1, we have in both cases α bounded or not, for $\varepsilon > 0$ small enough, we have as $x \to \infty$:

$$\frac{Lg(x,x+a)}{\ell''(a)\left(1 - \frac{S(x)}{S(x+a)}\right)} > 1 + \varepsilon.$$

And then we can conclude as in the proof of the proposition.

Although we do not need it, this also implies Proposition 4.2 (ii).

Then we examine the decreasing part of γ. Equation (5.13) is replaced by:

$$g(x(z),z) - g_x(x(z),z)\frac{S(x(z))}{S'(x(z))} - \ell(z) = 0. \tag{8.9}$$

Now in the proof of Proposition 5.2, the new ODE for the Cauchy problem is:

$$\gamma'(x) = \frac{Lg(x,\gamma)S(x)}{(\ell'(\gamma-x)-\ell'(\gamma))S'(x) + \ell''(\gamma-x)S(x)\left(1-\frac{S(x)}{S(\gamma)}\right)},$$

and for any x and γ, there exists $y \in (\gamma - x, \gamma)$ such that:

$$\begin{aligned}(\ell'(\gamma-x)-\ell'(\gamma))S'(x)+\ell''(\gamma-x)S(x)\left(1-\frac{S(x)}{S(\gamma)}\right) &= -x\ell''(y)S'(x)+\ell''(\gamma-x)S(x)\left(1-\frac{S(x)}{S(\gamma)}\right)\\ &\le -x\ell''(y)+\ell''(\gamma-x)S(x)\\ &\le \ell''(\gamma-x)(S(x)-xS'(x)).\end{aligned}$$

The last inequality follows from the fact that $\ell^{(3)} \geq 0$. Since $\ell''(x) > 0$ for $x > 0$, we can proceed as in the proof of Proposition 5.2. And finally Proposition 5.3 is replaced by the following:

Proposition 8.2 *We have one of the following cases:*
- $\gamma_\uparrow$ *is increasing on* $[0, +\infty)$*, and this implies* $\Gamma^0 < \Gamma^\infty$*;*
- $\gamma_\downarrow(x^*) = x^*$ *and* $x^* \geq \Gamma^\infty$*, which implies* $\Gamma^0 > \Gamma^\infty$*;*
- $|graph(\gamma_\downarrow) \cap graph(\gamma_\uparrow)| = 1$.
In the first case we write $\bar{x} = 0$ *and* $\bar{z} = \gamma_\uparrow(0)$*. In the second case we write* $\bar{x} = \bar{z} = x^*$*. In the third case, we write* $(\bar{x}, \bar{z}) = graph(\gamma_\downarrow) \cap graph(\gamma_\uparrow)$*. In all three cases we have* $(\bar{x}, \bar{z}) \in \Gamma^+$ *and* $\{(x, \gamma_\uparrow(x));\ x > \bar{x} \text{ and } \gamma_\uparrow(x) > x\} \subset Int(\Gamma^+)$*.*

Remark 8.1 In the second case of the previous proposition, the condition $x^* \geq \Gamma^\infty$ is not a priori a consequence of $\gamma_\downarrow(x^*) = x^*$, as there is no reason in general for the set $\mathrm{Int}(\Gamma^-)$ to be connected.

Finally, Theorem 6.1 can be proved in the same way in general, using the asymptotic expansions of Proposition 8.1.

We define v by:

$$v(x,z) = \ell(z) + g_x(\phi_\downarrow(z), z)\frac{S(x)}{S'(\phi_\downarrow(z))} \text{ if } (x,z) \in A_1 \tag{8.10}$$

$$v(x,z) = g(\phi_\uparrow(z), z) + g_x(\phi_\uparrow(z), z)\frac{S(x) - S(\phi_\uparrow(z))}{S'(\phi_\uparrow(z))} \text{ if } (x,z) \in A_2 \tag{8.11}$$

$$v(x,z) = \ell(z) + S(x)[\int_z^\infty \frac{\ell'(u)}{S(u)}du - K] \text{ if } (x,z) \in A_3 \tag{8.12}$$

$$v(x,z) = g(x,z) \text{ if } (x,z) \in A_4, \tag{8.13}$$

where $K = \int_{\bar{z}}^\infty \frac{\ell'(u)}{S(u)}du - \frac{g_x(\bar{x},\bar{z})}{S'(\bar{x})}$.

The proof of Lemmas 6.1 and 6.2 still work in this case, with the new definition of v, and the new equations for γ. In the proof of Proposition 6.1, we can use the asymptotic expansions of Proposition 8.1 in order to get $g_x(\phi_\uparrow(z), z) = O(1)$, $v(z,z) - g(\phi_\uparrow(z), z) = \circ(1)$ and $g(z,z) - g(\phi_\uparrow(z), z) = \circ(1)$, and the result follows. Finally, the proof of Proposition 6.2 still holds.

Appendix 1: Proof of Proposition 4.1

Proof. Recall that $\left(\frac{1}{\alpha}\right)' \to 0$ at infinity as stated in Remark 2.1-(ii). All the limits and equivalents are when $z \to +\infty$.

(i): As $S(z) \to +\infty$, $S(z) = \int_0^z e^{\int_0^u \alpha(v)dv} \sim \int_1^z e^{\int_0^u \alpha(v)dv}$. Integrating by parts, we get:

$$\int_1^z e^{\int_0^u \alpha(v)dv} = \left[\frac{e^{\int_0^u \alpha(v)dv}}{\alpha(u)}\right]_1^z - \int_1^z \left(\frac{1}{\alpha}\right)'(u)e^{\int_0^u \alpha(v)dv}du.$$

Since $\left(\frac{1}{\alpha}\right)' \to 0$, $\int_1^z \left(\frac{1}{\alpha}\right)'(u)e^{\int_0^u \alpha(v)dv}du = \circ\left(\int_1^z e^{\int_0^u \alpha(v)dv}\right)$, so that $S(z) \sim \frac{S'(z)}{\alpha(z)}$.

(ii): Using (i) and integrating by parts, we get:

$$\int_z^\infty \frac{du}{S(u)} \sim \int_z^\infty \frac{\alpha(u)}{S'(u)}du = \int_z^\infty \alpha(u)e^{-\int_0^u \alpha(v)dv}du \quad = \frac{1}{S'(z)};$$

$$\int_z^\infty \frac{udu}{S(u)} \sim \int_z^\infty \frac{u\alpha(u)}{S'(u)}du = \frac{z}{S'(z)} + \int_z^\infty \frac{1}{S'(u)}du.$$

But $u\alpha(u) \to \infty$ as $u \to \infty$, so that

$$\int_z^\infty \frac{1}{S'(u)}du = \circ\left(\int_z^\infty \frac{u\alpha(u)}{S'(u)}du\right),$$

and therefore:

$$\int_z^\infty \frac{udu}{S(u)} \sim \frac{z}{S'(z)}.$$

Finally, integrating by parts twice, we get:

$$\int_z^\infty \frac{u-z}{S(u)}du \sim \int_z^\infty \frac{(u-z)\alpha(u)}{S'(u)}du = \int_z^\infty \frac{1}{S'(u)}du$$
$$= \int_z^\infty \frac{\alpha(u)}{\alpha(u)S'(u)}du = \frac{1}{\alpha(z)S'(z)} + \int_z^\infty \left(\frac{1}{\alpha}\right)'(u)\frac{1}{S'(u)}du.$$

As $\left(\frac{1}{\alpha}\right)'(u) \to 0$ as $u \to \infty$, we get the result. □

Appendix 2: Proof of Proposition 8.1

Proof. (i): The proof is close to the proof of Proposition 4.1-(ii). First as φ is measurable and satisfies $0 \le \varphi(z) \le z$, the expressions make sense and the integrals exist. Then, using Proposition 4.1-(i) and integrating by parts, we have:

$$\int_z^\infty \frac{\ell''(u-\varphi(z))}{S(u)}du \sim \int_z^\infty \frac{\alpha(u)\ell''(u-\varphi(z))}{S'(u)}du = \int_z^\infty \alpha(u)e^{-\int_0^u \alpha(v)dv}\ell''(u-\varphi(z))du$$
$$= \frac{\ell''(z-\varphi(z))}{S'(z)} + \int_z^\infty \frac{\ell^{(3)}(u-\varphi(z))}{S'(u)}du.$$

According to assumption (8.4), all the terms above are non-negative. Moreover, using (8.5) we get:

$$\int_z^\infty \frac{\ell^{(3)}(u-\varphi(z))}{S'(u)}du \le K_1 \int_z^\infty \frac{\ell''(u-\varphi(z))}{S'(u)}du$$

while $$\int_z^\infty \frac{\alpha(u)\ell''(u-\varphi(z))}{S'(u)}du \ge \alpha(z)\int_z^\infty \frac{\ell''(u-\varphi(z))}{S'(u)}du \ (>0),$$

so that $A := \limsup_{z\to\infty} \frac{\int_z^\infty \frac{\ell^{(3)}(u-\varphi(z))}{S'(u)}du}{\int_z^\infty \frac{\alpha(u)\ell''(u-\varphi(z))}{S'(u)}du} < 1$, which means that, for z large enough, there exists a certain $k(z) \in \left[0, \frac{1+A}{2}\right)$ such that

$$\int_z^\infty \frac{\ell^{(3)}(u-\varphi(z))}{S'(u)}du = k(z)\int_z^\infty \frac{\alpha(u)\ell''(u-\varphi(z))}{S'(u)}du + \circ\left(\int_z^\infty \frac{\alpha(u)\ell''(u-\varphi(z))}{S'(u)}du\right).$$

As $\varphi(z) < z$ if $z > 0$, $\ell''(z-\varphi(z)) > 0$, and this implies that

$$(1-k(z))\int_z^\infty \frac{\alpha(u)\ell''(u-\varphi(z))}{S'(u)}du \sim \frac{\ell''(z-\varphi(z))}{S'(z)}.$$

Setting $\delta(z) = \frac{1}{1-k(z)} \in \left[1, \frac{2}{1-A}\right]$, we have the result. We also see that if $\alpha(x) \to \infty$ as $x \to \infty$, then $k(z) = 0$, so $\delta(z) = 1$.

(ii): Follows the lines of (i), replacing ℓ'' by ℓ', and using (8.6) instead of (8.5). □

Appendix 3: application to a hedging strategy

We have applied this result with the following strategy. Assume that X is an Ornstein-Uhlenbeck process with parameter α. We compute γ for $\ell(x) = \frac{x^2}{2}$. Assume at $t = 0$, $X_0 > 0$, then the first time $t \in [0, T_0]$ such that $X_t \geq \gamma(Z_t)$, we sell 1 stock of X. At $t = T_0$, we close the position. Then we reinitialize everything and do the same with the minimum (of course we buy instead of sell in this case).

We compare it to a family of strategies that we call "fixed barriers". We fix an a priori barrier level $b > 0$, and if $X > 0$, we sell 1 stock the first time $X_t \geq b$, then close the position at $t = T_0$, and do the symmetric if $X_t \leq -b$. We have tested those strategies in two cases. First a theoretical example, where we simulate the OU process X and use the "right" parameter α, then a market data example, where we took a process X computed from market data, assumed it behaved as an OU process and tried to estimate the parameter. More precisely in this second case, we took two stocks A and B, and computed:

$$X = \frac{\frac{A}{B}}{MA(\frac{A}{B})} - 1$$

where $MA(Y)$ is the moving average of Y (on a 3-month period).

We present hereafter the annualized Sharpe ratios obtained. What we call "a posteriori best barrier" is the best result we obtained with a fixed barrier b while b described $\mathbb{R}_+$, so there is no way to know how to fix it. In fact, in every simulation that we made, this "a posteriori best barrier" b_0 was close to Γ^0, which is not very surprising as the plot of γ for an OU process is quite flat. We emphasize on the fact that for a random barrier b, the Sharpe ratio is most of the time very small and can even be 0.

Data	Detection method	Sharpe ratio
Theoretical data	optimal stopping	2,1
	a posteriori best barrier	2
Market data	optimal stopping	1,8
	a posteriori best barrier	1,6

References

[1] M. Dai, H. Jin, Y. Zhong and X. Zhou (2009). Buy low and sell high. Working paper.

[2] J. Du Toit and G. Peskir (2007). The trap of complacency in predicting the maximum. *Ann. Probab.* 35, 340-365.

[3] J. Du Toit and G. Peskir (2008). Predicting the time of the ultimate maximum for Brownian motion with drift. *Proc. Math. Control Theory Finance* (Lisbon 2007), 95-112, Springer Berlin.

[4] J. Du Toit and G. Peskir (2009). Selling a stock at the ultimate maximum. *Ann. Appl. Probab.* 19 (3), 983-1014.

[5] S.E. Graversen, G. Peskir and A.N. Shiryaev (2001). Stopping Brownian motion without anticipation as close as possible to its ultimate maximum. *Theory Probab. Appl.* 45 (125-136).

[6] D. Hobson (2007). Optimal stopping of the maximum process: a converse to the results of Peskir. *Stochastics* 79 (1), 85-102.

[7] S. Karlin and H.M. Taylor (1981). *A Second Course in Stochastic Processes.* Academic Press.

[8] R. Myneni (1992). The pricing of American option. *Ann. Appl. Probab.* Vol. 2, No. 1 (1-23).

[9] J. Obloj (2007). The maximality principle revisited: on certain optimal stopping problems. *Séminaire de Probabilités XL, Lect. Notes in Math.* 1899, 309-328.

[10] J.L. Pedersen (2003). Optimal prediction of the ultimate maximum of Brownian motion. *Stoch. Stoch. Rep.* 75 (205-219).

[11] G. Peskir (1998). Optimal stopping of the maximum process: the maximality principle. *Ann. Probab.* Vol. 26, No. 4 (1614-1640).

[12] G. Peskir and A.N. Shiryaev (2006). *Optimal Stopping and Free-Boundary Problems.* Lectures in Mathematics, ETH Zurich, Birkhauser.

[13] L.A. Shepp and A.N. Shiryaev (1993). The Russian option: reduced regret. *Ann. Appl. Probab.* 3 (631-640).

[14] L.A. Shepp and A.N. Shiryaev (1994). A new look at the Russian option. *Theory Probab. Appl.* 39 (103-119).

[15] A.N. Shiryaev (1967). Two problems of sequential analysis. *Cybernetics* 3, No.2 (79-86).

[16] A.N. Shiryaev (1978). *Optimal Stopping Rules.* Springer-Verlag, Berlin-New York.

[17] A.N. Shiryaev (2002). Quickest detection problems in the technical analysis of the financial data. *Proc. Math. Finance Bachelier Congress* (Paris, 2000), Springer (487-521).

[18] A.N. Shiryaev, Z. Xu and X. Zhou (2008). Thou shalt buy and hold. *Quantitative Finance* 8, 765-776.

[19] M.A. Urusov (2005). On a property of the moment at which Brownian motion attains its maximum and some optimal stopping problems. *Theory Probab. Appl.* 49 (169-176).

[20] H. Zhang and Q. Zhang (2008). Trading a mean-reverting asset: buy low and sell high. *Automatica* 44 (1511-1518).