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Optimal Pits and Optimal Transportation

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Abstract

In open pit mining, one must dig a pit, that is, excavate the upper layers of ground before reaching the ore. The walls of the pit must satisfy some mechanical constraints, in order not to collapse. The question then arises how to mine the ore optimally, that is, how to find the optimal pit. We set up the problem in a continuous (as opposed to discrete) framework, and we show, under weak assumptions, the existence of an optimum pit. For this, we formulate an optimal transportation problem, where the criterion is lower semi-continuous and is allowed to take the value $+\infty$. We show that this transportation problem is a strong dual to the optimum pit problem, and also yields optimality (complementarity slackness) conditions.

All references are to [1]The data:

- A compact subset $E \subset \mathbb{R}^3$
- A continuous function $g : E \to \mathbb{R}$
- A compact-valued map $\Gamma : E \twoheadrightarrow E$, such that:

$$\begin{array}{rrcl} & \text{(reflexivity)} & & z \in \Gamma(z) \\ \text{(transitivity)} & [z_2 \in \Gamma(z_1) \text{ and } z_3 \in \Gamma(z_2)] & \Longrightarrow & z_3 \in \Gamma(z_1) \end{array}$$

Write $z_2 \succeq z_1$ for $z_2 \in \Gamma(z_1)$. It is a partial ordering of E. A subset $A \subset E$ is *stable* if $z \in A$ implies that $\Gamma(z) \subset A$. The family of all compact stable subsets of E will be denoted by $\mathcal{S}(E)$:

$$A \in \mathcal{S}(E) \Longleftrightarrow A = \cup_{z \in A} \Gamma(z)$$

The interpretations are as follows: E is the region (up to the surface) containing the ore. The relation $z_2 \in \Gamma(z_1)$ means that one must extract z_2 before extracting z_1, and g is the net profit obtained by extracting dz at z, once it has become accessible. Any mining profile $A \subset E$ has to be stable, so that all the ore in A can be extracted, and the corresponding profit is $\int_A g(z)\, dz$ We are looking for the profile that maximizes profit, that is, we are trying to solve the optimization problem:

$$\begin{array}{c} \max \int g(z)\, dz \\ A \in \mathcal{S}(E) \end{array} \tag{1}$$

We shall set it up as an optimal transportation problem. Assume the following:

Condition 1 *E is stable, the graph of Γ is closed and $g(z) > 0$ for some $z \in E$*

Introduce the following subsets of E

$$\begin{aligned} E^+ &:= \overline{\{g(x) > 0\}} \\ E^- &:= \overline{\{g(x) < 0\}} \end{aligned}$$

Both E^+ and E^- are compact sets. We endow them with the measure $|g(z)|\, dz$, so that the subsets $\{g(z) = 0\}$ have measure zero. Introduce a source α and a sink ω, and set:

$$X = E^+ \cup \{\alpha\}, \quad Y = E^- \cup \{\omega\}$$

Both X and Y are compact sets. We endow them with the measures μ and ν defined by:

$$\begin{array}{c} \mu(\{\alpha\}) = \int_{E^-} |g(z)|\, dz, \quad \mu|_{E^+} = g(z)\, dz \\ \nu(\{\omega\}) = \int_{E^+} g(z)\, dz, \quad \nu|_{E^-} = |g(z)|\, dz \end{array}$$

so that $\mu(X) = \nu(Y)$.

Define the cost $c : X \times Y \longrightarrow R$ as follows:

X	Y	$c(x, y)$
$x \in E^+$	$y \in \Gamma(x)$	0
$x \in E^+$	$y \notin \Gamma(x),\ y \in E^-$	$+\infty$
$x \in E^+$	$y = \omega$	1
$x = \alpha$	$y \in Y$	0

Note that, because Γ has closed graph, c is lower semi-continuous (l.s.c.)

Let π be a positive measure on $X \times Y$, and let π_X and π_Y be its marginals. Denote by $\Pi(\mu, \nu)$ the set of all Radon probability measures such that $\pi_X = \mu$ and $\pi_Y = \nu$. Now consider the optimal transportation problem in Kantorovitch form:

$$\begin{array}{c} \min \int_{X \times Y} c(x, y)\, d\pi \\ \pi \in \Pi(\mu, \nu) \end{array} \tag{K}$$

where π is a positive measure, and π_X and π_Y its marginals. Write the conditional probabilities as P_x and P_y, so that:

$$\pi = \int_X P_x d\mu = \int_Y P_y d\nu$$

Theorem 2 *The minimum of problem (K) is attained. Assume that it is positive, finite, and attained at* π *:*

$$\int_{X\times Y} c(x,y)\,d\pi > 0$$

Let $B = \{x \mid P_x\{(\omega)\} > 0\}$*. Then* $A := \cup_{x\in B}\Gamma(x)$ *is the solution of problem (P)*

If the minimum is finite, then $P_x(\Gamma(x)\cup\{\omega\}) = 1$ for almost every x, so xdx is sent either to $\Gamma(x)$ (at cost 0) or to ω (at cost 1). Clearly, because of the minimization, one will not start transporting mass at cost 1 until all the possibilities of transporting at cost 0 have been exhausted: one will never send mass at ω (that is, one will never have $P_{\bar{x}}(\{\omega\}) > 0$) unless $\int_{E^+} P_x(\Gamma(\bar{x}))\,d\mu(x) = \mu(\Gamma(\bar{x}))$ (that is, one has extracted all of $\Gamma(\bar{x})$).

Of course, all this has to be made rigorous. In order to to this, we introduce the Kantorovitch dual of problem (K). Set:

$$\begin{aligned} \mathcal{A} \; &: = \left\{(p,q) \mid p\in L^1(d\mu),\ q\in L^1(d\nu),\ p(x)-q(y)\leq c(x,y) \quad (\mu,\nu)\text{-a.s.}\right\} \\ J(p,q) \; &: = \int_X p d\mu - \int_Y q d\nu \end{aligned}$$

Consider the optimisation problem:

$$\begin{array}{c} \sup J(p,q) \\ (p,q)\in\mathcal{A} \end{array} \tag{2}$$

The Kantorovitch duality result (see Theorem 1.3) states that:

$$\begin{aligned} \int_{X\times Y} c(x,y)\,d\pi \quad &\leq \quad J(p,q) \quad \forall\pi\in\Pi(\mu,\nu),\ \forall(u,v)\in\mathcal{A} \\ \inf(\mathrm{K}) \quad &= \quad \sup(\mathrm{D}) \end{aligned}$$

Proposition 3 *Problem (K) has a solution*

Proof. The set of positive Radon measures on the compact space $X\times Y$ is weak-* compact, and the map $\pi \to \mathrm{E}^{\pi}[c]$ is weak-* l.s.c, so the result follows. ∎

Proposition 4 *Problem (D) has a solution*

Proof. Because $\mu(X) = \nu(Y)$, there is a translation-invariance buit into the problem: $J(p+a, q+a) = J(p,q)$ for all constants a. So, without loss of generality, we may assume that $q(\omega) = 0$.

Note that $u_1 \leq u_2$ implies that $J(u_1,v) \leq J(u_2,v)$. Take a maximizing sequence $(p_n,q_n)\in L^1(\mu)\times L^1(\nu)$, $q_n(\omega) = 0$. We get another maximizing sequence by setting:

$$
\begin{aligned}
p_n(x) &= \min\{1+q_n(\omega),\ \inf\{q_n(y)\ \mid y\in\Gamma(x)\}\} && (3)\\
p_n(\alpha) &= \min\left\{q_n(\omega),\ \inf\left\{q_n(y)\ \mid y\in E^-\right\}\right\} && (4)\\
q_n(y) &= \max\{p_n(\alpha), \sup\{p_n(x)\ \mid y\in\Gamma(x)\}\} && (5)\\
q_n(\omega) &= \max\left\{p_n(\alpha), \sup\left\{p_n(x)-1\mid x\in E^+\right\}\right\} && (6)
\end{aligned}
$$

It follows from (3), (4) and $q_n(\omega)=0$ that $p_n(x)\le 1$ for $x\in E^+$ and $p_n(\alpha)\le 0$. Writing this in (6), we find that there are two cases (after extracting a subsequence): either $p_n(\alpha)=q_n(\omega)=0$ for all n, or $\sup_{x\in E^+} p_n(x)=1$.

If $p_n(\alpha)=0$ for all n, we get from (4) and $q_n(\omega)=0$ that $q_n(y)\ge 0$ for $y\in E^-$. Writing this in (3), with $p_n(\alpha)=0$,we find that $p_n(x)\ge 0$ for $x\in E^+$. Since we already know that $p_n(x)\le 1$, and $p_n(\alpha)=0$, we conclude that $0\le p_n\le 1$ on X. Similarly, we find from (5) that $0\le q_n\le 1$ on Y. So the family (p_n,q_n) is equi-integrable in $L^1(\mu)\times L^1(\nu)$. By the Dunford-Pettis theorem, we can extract a subsequence which converges weakly to some (p,q). Since the admissible set $\mathcal{A}$ is convex and closed, it is weakly closed, and $(p,q)\in\mathcal{A}$. Since J is linear and continuous, we get:

$$
J(p,q)=\lim_n J(p_n,q_n)=\sup_{\mathcal{A}} J
$$

so that $(p,q)\in\mathcal{A}$ is an optimal solution.

If $p_n(\alpha)<0$ for all n, then $\sup_{x\in E^+} p_n(x)=1$. ■

References

[1] Villani, Topics in Optimal Transportation, Graduate Studies in Mathematics vol. 58, AMS 2003