Asset liability management under solvability constraints

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Motivation and introduction

- Consider an energy company operating nuclear power plants;
- regulation requires the company to respect a solvability constraint with respect to the long-term costs of decommissioning and dismantling nuclear power plants;
- the regulator monitors over time the company's solvability.

In this talk:

- A continuous-time asset-liability management problem;
- a solvability constraint that is continuously monitored with respect to an exogenous liability process, representing the costs mentioned above;
- mathematically, a mixed regular/singular stochastic control problem:
  - how to optimally allocate funds in the financial market?
  - what is the cheapest way of ensuring that the liability is met?

Notation and model description

- Let $P = (P_t)_{t \geq 0}$ be the liability process, assumed to be an increasing adapted process with $P_0 = 1$.
- We consider a simple financial market with a single risky asset, whose price satisfies the Black-Scholes dynamics

$$dS_t = S_t(\mu dt + \sigma dW_t), \qquad S_0 > 0,$$

  with $\mu > 0$ and $\sigma > 0$.
- A strategy is a triplet (ψ, C^+, C^-), where:
  - $\psi = (\psi_t)_{t \geq 0}$ represents the position in the risky asset,
  - $C^+ = (C_t^+)_{t \geq 0}$ is the cumulated capital injected into the portfolio,
  - $C^- = (C_t^-)_{t \geq 0}$ is the cumulated capital withdrawn from the portfolio.

  It is assumed that $C^{\pm} \in \mathcal{C}$, with $\mathcal{C}$ denoting the set of all càglàd adapted increasing processes with $C_0^{\pm} = 0$.

Wealth dynamics and admissible strategies

- The wealth process $X := X^{x,\psi,C^+,C^-}$ associated with a strategy $(\psi, C^+, C^-) \in L^2_{loc}(W) \times \mathcal{C} \times \mathcal{C}$ and initial wealth x is given by

$$X_t = x + \int_0^t \psi_u \, dS_u + C_t^+ - C_t^-, \qquad \text{for all } t \geq 0.$$

- A triplet $(\psi, C^+, C^-) \in L^2_{loc}(W) \times \mathcal{C} \times \mathcal{C}$ is *admissible* starting from the initial wealth x if $(\psi, C^+, C^-) \in \mathcal{A}_x(P)$, where

$$\mathcal{A}_x(P) := \left\{(\psi, C^+, C^-) \in L^2_{loc}(W) \times \mathcal{C} \times \mathcal{C} : X_t \geq P_t \ \mathbb{P}\text{-a.s., } \forall t \geq 0\right\}.$$

The general optimization problem

Our general goal is to solve

$$\inf_{(\psi, C^+, C^-) \in \mathcal{A}_x(P)} \mathbb{E}\left[\int_0^\infty e^{-\theta s} f(C_s^+, C_s^-) ds\right] =: V(x),$$

where $f : \mathbb{R}_+^2 \to \mathbb{R}$ is strictly increasing (decreasing, resp.) in the first (second, resp.) argument and $\theta > 0$ is a discount factor.

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Lemma

For any $x > 0$, it holds that

$$V(x) = \inf_{(\psi, C^-)\in L^2_{loc}(W)\times \mathcal{C}} \mathbb{E}\left[\int_0^\infty e^{-\theta s} f(C_s^{+,*}, C_s^-)ds\right],$$

where

$$C_t^{+,*} = \sup_{0\le s\le t}\left\{\left(P_s - x - \int_0^s \psi_u dS_u + C_s^-\right)\vee 0\right\}, \qquad \textit{for all } t \ge 0.$$

⇒ capital is being injected only when "there is no other choice".

Problem formulation

- Consider an exponentially growing deterministic liability process P:

$$P_t = e^{\delta t}, \qquad \text{for all } t \geq 0 \text{ and a growth rate } \delta > 0.$$

- Trading strategies $\phi = (\phi_t)_{t \geq 0}$ are now parameterized in terms of proportion of wealth invested in the risky asset, with the constraint

$$\phi_t \in [0, 1] \quad \mathbb{P}\text{-a.s., for all } t \geq 0,$$

representing *no short-selling or leveraged positions*.
Let Φ be the set of all such trading strategies ϕ and, as above,

$$\mathcal{A}_x(P) := \{(\phi, C^+, C^-) \in \Phi \times \mathcal{C} \times \mathcal{C} : X_t \geq P_t \ \mathbb{P}\text{-a.s., } \forall t \geq 0\}.$$

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- We now aim at studying the following optimization problem:

$$\inf_{(\phi, C^+, C^-) \in \mathcal{A}_x(P)} \mathbb{E}\left[\int_0^{\infty} e^{-\theta s} (C_s^+ - \lambda C_s^-) ds \right], \qquad (1)$$

for some $\lambda \in (0,1)$, measuring the benefit of withdrawing capital.

A reformulation of the problem

We recast the model into a time-homogeneous setting:

- for $(\phi, C^+, C^-) \in \mathcal{A}_x(P)$, let, for all $t \geq 0$,
$$\widetilde{X}_t := e^{-\delta t} X_t \quad \text{and} \quad \widetilde{C}_t^{\pm} := e^{-\delta t} C_t^{\pm};$$
in addition, for $C^{\pm} \in \mathcal{C}$, let define the process $L^{\pm} = (L_t^{\pm})_{t \geq 0}$ by
$$L_t^{\pm} := \int_0^t e^{-\delta s} dC_s^{\pm};$$
- the processes $\widetilde{X} = (\widetilde{X}_t)_{t \geq 0}$ and $\widetilde{C}^{\pm} = (\widetilde{C}_t^{\pm})_{t \geq 0}$ satisfy
$$d\widetilde{X}_t = -\delta \widetilde{X}_t dt + \phi_t \widetilde{X}_t (\mu dt + \sigma dW_t) + dL_t^+ - dL_t^-,$$
$$d\widetilde{C}_t^{\pm} = -\delta \widetilde{C}_t^{\pm} dt + dL_t^{\pm};$$
- problem (1) can be then rewritten as
$$\inf_{(\phi, L^+, L^-) \in \mathcal{A}_x(1)} \mathbb{E}\left[\int_0^\infty e^{-(\theta-\delta)s} (\widetilde{C}_s^+ - \lambda \widetilde{C}_s^-) ds\right]. \qquad (2)$$

Preliminary properties

To problem (2), we can associate the value function

$$v(c^+, c^-, x) := \inf_{\phi, L^+, L^-} \mathbb{E}\left[\int_0^\infty e^{-(\theta-\delta)s}(\widetilde{C}_s^+ - \lambda \widetilde{C}_s^-)\, ds \,\middle|\, \widetilde{C}_0^\pm = c^\pm, \widetilde{X}_0 = x\right].$$

By the linear structure of our minimization criterion, it can be show that

$$v(c^+, c^-, x) = (c^+ - \lambda c^-)/\theta + v(0, 0, x).$$

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Lemma

1. *The function $g(x) := v(0,0,x)$ is convex.*
2. *If $\theta > \delta$, then, for all $x \geq 1$, it holds that*

$$g(x) \leq \frac{\delta}{\theta(\theta-\delta)} x^{1-\frac{\theta}{\delta}} \mathbf{1}_{\{x \leq \lambda^{-\delta/\theta}\}} + \left(\frac{\lambda^{1-\frac{\delta}{\theta}}}{\theta-\delta} - \frac{\lambda}{\theta} x\right) \mathbf{1}_{\{x > \lambda^{-\delta/\theta}\}}.$$

3. *If $\theta > \delta$ and $\mu > \theta$, then $g(x) = -\infty$, for every $x \geq 1$.*

The bound is what you get if you do not invest in the risky asset.

The HJB variational inequality

By a formal application of Itô's formula and the dynamic programming principle, the value function $v(c^+, c^-, x)$ satisfies

$$\min\Big\{\inf_{\phi\in[0,1]}\Big(-(\theta-\delta)v+c^+-\lambda c^-+\frac{\sigma^2}{2}\phi^2x^2v_{xx}+(\mu\phi-\delta)xv_x \\ -\delta c^+v_{c^+}-\delta c^-v_{c^-}\Big);v_x+v_{c^+};-v_x+v_{c^-}\Big\}=0.$$

In turn, the decomposition $v(c^+, c^-, x) = (c^+ - \lambda c^-)/\theta + g(x)$ leads to

$$\inf_{\varphi\in[0,1]}\left(-(\theta-\delta)g(x)+\frac{\sigma^2}{2}\varphi^2x^2g''(x)+(\mu\varphi-\delta)xg'(x)\right)\geq 0,$$

subject to the boundary conditions

$$g'(x)\geq -1/\theta \qquad \text{and} \qquad g'(x)\leq -\lambda/\theta,$$

for all $x \geq 1$, with at least one equality.

The value function without capital withdrawal

Let us consider the limiting case where $\lambda \to 0$, meaning that we do not receive any benefit from withdrawing capital.

In this case, the above problem admit an explicit and easy solution.

Proposition

Suppose that $\theta > \delta$. *Then the function*

$$g(x) = -\frac{1}{\theta(1+\eta)} x^{1+\eta}$$

solves the above HJB variational inequality, where $\eta < -1$ *is given by*

$$\eta = \begin{cases} \eta^a := \frac{-\theta - \frac{\mu^2}{2\sigma^2} - \sqrt{(\theta + \frac{\mu^2}{2\sigma^2})^2 - 2\delta\frac{\mu^2}{\sigma^2}}}{2\delta}, & \text{if } \mu \leq -\eta^a\sigma^2; \\ \eta^b := -\frac{1}{2} + \frac{\delta-\mu}{\sigma^2} - \frac{1}{\sigma^2}\sqrt{\left(\frac{\sigma^2}{2} + \mu - \delta\right)^2 + 2\sigma^2(\theta-\mu)}, & \text{otherwise.} \end{cases}$$

The optimal strategy

The (candidate) optimal strategy $(\phi^*)_{t\geq 0}$ is given by the constant proportion strategy

$$\phi_t^* = -\frac{\mu}{\sigma^2 \eta^a} \wedge 1, \qquad \text{for all } t \geq 0, \tag{3}$$

and the (candidate) optimal capital injection process $(C^{+,*})_{t\geq 0}$ is given by

$$C_t^{+,*} = \int_0^t e^{\delta s} dL_s^{+,*},$$

where $L_t^{+,*} = \sup_{0\leq s\leq t}\{Y_s(x,\phi^*) \vee 0\}$, with

$$Y_s(x,\phi^*) := -\log x + \delta s - \phi^* \left(\mu - \frac{\sigma^2}{2}\phi^*\right) s - \phi^* \sigma W_s.$$

A verification theorem allows to show that $(\phi^*, C^{+,*})$ is indeed the optimal strategy and the value function is as given in the previous proposition.

Conclusion and outloook

- A mixed regular/singular stochastic control problem motivated by ALM under solvability constraints.
- Fully explicit solution in the case of exponentially growing liabilities, linear performance functional and no withdrawals.
- Explicit solution when allowing for withdrawing capital?
  We know that it is optimal to inject capital only when we are forced to.
  Conjecture:
  - there exists an endogenous "safety threshold" $x^* > 1$ such that we withdraw capital as soon as our wealth exceeds x^*;
  - optimal capital processes $(C^{+,*}, C^{-,*})$ given as the solution to a double Skorokhod reflection problem on $[1, x^*]$.
- What about alternative price processes for the risky asset (e.g., regime switching or benchmark models)?

Thank you for your attention