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HJB Equations and Extensions of Classical Stochastic Control Theory

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HJB EQUATIONS AND EXTENSIONS OF CLASSICAL STOCHASTIC CONTROL THEORY

NOTES FROM P.-L. LIONS' LECTURES AT THE COLLÈGE DE FRANCE

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ABSTRACT. These notes are a transcription of the course *"Équations de HJB et extensions de la théorie classique du contrôle stochastique"*, given by P.-L. Lions at the Collège de France in 2016/2017. The course contains a few developments on Bayesian learning, on optimal control of conditioned processes, on interfaces and junction problems, and a seminar on scalar conservation laws which was presented by P.-L. Lions as a part of his course.

CONTENTS

1. BEYOND STOCHASTIC CONTROL

1.1. **Bayesian control.** This type of problems are motivated by some questions arising from the field of advertising. The Bayesian control theory is a different formulation of optimal control problems with partial information. The equations resulting from these problems has already been studied in the past, however the techniques were not developed enough at the time to handle HJB equations in infinite dimension. The Mean Field Games (MFG for short) theory has allowed to develop many tools for infinite dimensional

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HJB equations. We revisit classical problems of stochastic control, by trying to improve some results using this recent progress.

We begin the presentation with an example called Bayesian learning. We consider the following dynamics

$$\mathrm{d}X_t = a\alpha_t\,\mathrm{d}t + \mathrm{d}W_t, \quad X_0 = x \in \mathbb{R}^d.$$

The noise is modeled by a standard $\mathbb{R}^d$-Brownian motion W_t, a is an unknown parameter and α_t is the control. In a classical optimal control situation, if the noise and α are known, then we know X, and the whole problem consists in choosing the right control. In this situation there is a lack of information related to the parameter a.

Let μ_t be a probability measure which expresses the lack of information related to the parameter a at the instant $t \geq 0$. We start with an initial probability measure (an initial assumption) μ_0. Having observed X_t up to the moment t, we want to learn more and better the values of the parameter a through the evolution μ_t of the initial probability measure μ_0. The question that naturally arises is the following : starting from an observation of $\mathrm{d}X_t$, how to build the measure $\mu_{t+\mathrm{d}t}$ (the new hypotheses on a). One have

$$\mu_{t+dt}(a) = \frac{\mu_t(a)\exp(-(\,\mathrm{d}X_t - a\alpha_t\,\mathrm{d}t)^2/2\,\mathrm{d}t)}{\int \mu_t(b)\exp(-(\,\mathrm{d}X_t - b\alpha_t\,\mathrm{d}t)^2/2\,\mathrm{d}t)\,\mathrm{d}b}.$$

Moreover, we formally assume that $(DX_t)^2$ is of an order dt, so that up to terms of order "$\mathrm{d}t$" we have

$$\mu_{t+dt}(a) = \frac{\mu_t(a)\exp(a\alpha_t\,\mathrm{d}X_t - a^2\alpha_t^2\,\mathrm{d}t/2)}{\int \mu_t(b)\exp(b\alpha_t\,\mathrm{d}X_t - b^2\alpha_t^2\,\mathrm{d}t/2)\,\mathrm{d}b},$$

thanks to the fact that the increments of the Brownian motion are independent. Using an asymptotic expansion, we infer that

$$\mu_{t+dt}(a) = \mu_t(a)\frac{1 + a\alpha_t\,\mathrm{d}X_t - a^2\alpha_t^2\,\mathrm{d}t/2 + a^2\alpha_t^2(\,\mathrm{d}X_t)^2/2}{1 + \int b\mu_t\alpha_t\,\mathrm{d}X_t},$$

so that

$$\begin{aligned}\mathrm{d}\mu_t \;=\; & \mu_t(a)\left\{\left(a\alpha_t\,\mathrm{d}X_t - \left(\int b\mu_t(b)\,\mathrm{d}b\right)\alpha_t\,\mathrm{d}X_t\right) + \left(\int b\mu_t(b)\,\mathrm{d}b\right)^2\alpha_t^2\,\mathrm{d}t\right\}\\ & -a\alpha_t^2\mu_t(a)\left(\int b\mu_t(b)\,\mathrm{d}b\right)\mathrm{d}t,\end{aligned}$$

where the last expansion is pushed to the second order since there is a nonlinear operation. This explains why the other two additional terms are obtained. We infer the following problem,

$$\mathrm{d}\mu_t = \mu_t\left(a - \int b\mu_t\right)\alpha_t\,\mathrm{d}X_t - \left(\int b\mu_t\right)\left(a - \int b\mu_t\right)\alpha_t^2\mu_t\,\mathrm{d}t, \quad \mu_0 \in \mathcal{P}(\mathbb{R}^d), \tag{1.1}$$

where $\int b\mu_t = \int b\mu_t(b)\,\mathrm{d}b$. Note that, this equation describes only the learning problem and not the optimisation mechanism. Moreover, note that all the parameters are coupled through mean values. Equation (1.1) is a stochastic partial differential equation, where X is a semi-martingale whose quadratic variation is that of a Brownian motion.

The existence of a learning process means that μ_t must reveal the speed mean value of X_t as $t \to \infty$. Therefore we hope that μ_t converges to a Dirac mass as $t \to \infty$. Let

us start with a first observation on the structure of equation (1.1): since equation (1.1) is multiplicative, the support of the measure μ is always decreasing. In particular, for any $t \geq 0$,

$$supp(\mu_t) \subset supp(\mu_0).$$

This shows the importance of the choice of the initial hypothesis. Indeed, if one chooses a bad support, one can never leave it and so, never learn the correct information.

In addition to the learning question, it is also important to understand how to learn quickly. More precisely, we are also interested in the relationship between the level of effort (which is reflected in the optimization problem) and the speed of the learning process. This would make a compromise between the effort required to learn quickly and the cost of this effort.

Note that in the derivation of equation (1.1), the terms that appears in the right hand side are essentially due to the normalization. One can therefore look at the evolution by eliminating all the terms that appeared because of the division. In fact, by removing the normalization we get much simpler equations. Let

$$\mathrm{d}\nu_t = \nu_t \alpha a \,\mathrm{d}X, \quad \nu_0 = \mu_0, \tag{1.2}$$

so that

$$\nu_t = \exp\left(a\int_0^t \alpha_s \,\mathrm{d}X_s - \frac{1}{2}a^2\int_0^t \alpha_s^2 \,\mathrm{d}s\right)\nu_0.$$

Now, set

$$m_t := \nu_t \Big/ \int \nu_t(b)\,\mathrm{d}b.$$

We have

$$\mathrm{d}m_t = \frac{\mathrm{d}\nu_t}{\int \nu_t} - \frac{\nu_t \,\mathrm{d}\int \nu_t}{(\int \nu_t)^2} - \frac{\mathrm{d}<\nu_t, \int \nu_t>}{(\int \nu_t)^2} + \frac{\nu_t <\mathrm{d}\int \nu_t>}{(\int \nu_t)^3},$$

so after computation one recovers from this expression equation (1.1). So, if there is uniqueness for equation (1.1), thus

$$\mu_t = \nu_t \Big/ \int \nu_t$$

and we can exploit the simplicity of equation (1.2). In both cases, we have an evolution in an infinite dimension space (which is $\mathcal{P}(\mathbb{R}^d)$ for equation (1.1), and $m_b^+(\mathbb{R}^d)$: the space of positive bounded measures, for equation (1.2)).

We shall start by looking at two simple examples where the reduction to finite dimension is possible.

Example 1.1 (Gaussian distribution). Assume that $\mu_0(\,\mathrm{d}\alpha) = \exp(-\lambda_0(a - A_0)^2/2)\,\mathrm{d}\alpha$, then

$$\nu_t = \exp\left(a\alpha X_t - \frac{1}{2}a^2\alpha^2 t\right)\nu_0$$

is a Gaussian with respect to a, and to compute its parameters, we rewrite it in the form

$$\nu_t = C\exp(-\lambda_t(a - A_t)^2/2).$$

Identifying the terms, it follows that

$$\lambda_t/2 = \lambda_0/2 + \alpha^2 t/2, \quad \text{and} \quad A_t\lambda_t = \lambda_0 A_0 + \alpha X_t,$$

so that

$$\mathrm{d}\lambda_t = \alpha^2 \,\mathrm{d}t. \tag{1.3}$$

Converging to a Dirac mass implies that the Gaussian is tightened, which means that $\lambda_t \to \infty$ when $t \to \infty$. Note that equation (1.3) implies that the concentration towards the Dirac mass is increasingly fast when α is large. Moreover, we have that

$$\mathrm{d}A_t = (\alpha_t/\lambda_t)\,\mathrm{d}X_t - (\alpha X_t/\lambda_t^2)\,\mathrm{d}\lambda_t.$$

Note that, in the last equation, it is not clear why the correct center is recovered.

Example 1.2 (Discrete distributions). Assume that the initial assumption is given by

$$\mu_0 = \sum_{i=1}^{k} \mu_i \delta_{a_i},$$

so that one can write μ_t and ν_t on the following form,

$$\mu_t = \sum_{i=1}^{k} \mu_{i,t}\delta_{a_i}, \quad \text{and} \quad \nu_t = \sum_{i=1}^{k} \nu_{t,i}\delta_{a_i}.$$

In this case, we simply have

$$\mathrm{d}\nu_{t,i} = \nu_{t,i}\alpha_t a^i \,\mathrm{d}X_t \quad \text{for any } 1 \le i \le k.$$

We reduce the problem to a problem of finite dimension k.

In these two particular cases the evolution of ν_t and μ_t is reduced to a finite-dimensional evolution. We shall now start look at the learning issue : do we have a learning process? What happens if we make bad assumptions initially?

1.1.1. *Do we have a learning process ?* Assume that $\alpha_t \equiv \alpha \neq 0$ is a constant and consider the following dynamics,

$$X_t = \alpha \bar{A} t + W_t + x.$$

By equation (1.1) we have that

$$\mu_t = \frac{\mu_0 \exp(a\alpha X_t - 1/2a^2\alpha^2 t)}{\int \mu_0(b)\exp(b\alpha X_t - 1/2b^2\alpha^2 t)\,\mathrm{d}b},$$

so that

$$\mu_t = \frac{\mu_0 \exp(-1/2(a-\bar{A})^2\alpha^2 t + a\alpha W_t)}{\int \mu_0(b)\exp(-1/2(b-\bar{A})^2\alpha^2 t + b\alpha W_t)\,\mathrm{d}b}.$$

We are interested in long-time behavior. Given that $W_t/t \to 0$ a.s, the t-terms will be preponderant when $t \to +\infty$. The following Lemma gives the main results on the long time behavior.

Lemma 1.3. *If $\bar{A} \in supp(\nu_0)$ then we have that $\mu_t \rightharpoonup \delta_{\bar{A}}$ when $t \to +\infty$. In general, $Supp(\mu_t)$ concentrates on*

$$\mathrm{argmin}\{|a - \bar{A}| \quad | \quad a \in Supp\mu_0\}.$$

In the case of one dimension, we have two possibilities :

- if there exists a unique minimum point denoted A_∞ then $\mu_t \rightharpoonup \delta_{A_\infty}$. (We converge to the nearest point of the support. We may have made incorrect assumptions, but we try to correct them at best.);

- otherwise, if we asumme $\bar{A} = 0$ (for simplicity) and start for instance with

$$\nu_0 = \theta\delta_{-a_0} + (1-\theta)\delta_{a_0}$$

where $a_0 > 0$, then μ_t converges in law to $1/2\delta_{-a_0} + 1/2\delta_{a_0}$. The convergence in law is in the sense that μ_t is a stochastic measure. In other words, it is the convergence in law on measures;

The above convergence result does not inform us about the learning speed and relationship with effort. Note also that this result illustrates the strong dependence of the asymptotic behavior on the initial distribution. In fact, in the last example, a slight modification of the position of the two points changes completely the result.

Example:

(1) Assume that $\mu_0 = \sum_{i=1}^k \theta_i \delta_{a_i}$, where $0 < \theta_i < 1$, and $|A - a_i| = \rho$. In this particular case, the convergence depends on the sign of the scalar product $\langle a.W_t \rangle$, so the result will depend on the directions of W_t. We infer that the result will depend strongly on the distribution of points on the sphere. In that case, one has a convergence in law towards

$$\sum_{i=1}^k \theta_i^\infty \delta_{a_i},$$

where θ_i^∞ is defined geometrically according to the proportions of angles that ensures the right directions.

(2) Assume that the initial distribution is given by $\mu_0 = \delta_{\partial B(A,\rho)}$. In this case, and by rotation invariance, we converge in law to the same initial distribution, i.e. $\delta_{\partial B(A,\rho)}$.

Remark 1.4 (on learning speed). Assume that we have a Gaussian distribution with $\alpha_t \equiv \alpha \neq 0$. For simplicity, we do the computations in the one dimensional case. The final results hold for higher dimensions. Assume that

$$\nu_0 = \frac{e^{-(a-A_0)^2/2\lambda_0}}{(2\pi/\lambda_0)^{1/2}}.$$

In this case ν_t is a Gaussian characterized by its mean and its variance. Using the computations of Example 1.1, it follows that

$$\lambda_t = \lambda_0 + \alpha^2 t.$$

Hence, the parameter α^2 characterizes the speed of concentration of the measure. In addition, according to Example 1.1 we have that

$$A_t\lambda_t = \alpha X_t + A_0\lambda_0,$$

so that,

$$A_t = \frac{A_0\lambda_0}{\lambda_t} + \frac{\alpha^2 A t}{\lambda_t} + \frac{\alpha W_t}{\lambda_t}.$$

Hence,

$$A_t = A + \frac{\lambda_0(A_0 - A)}{\lambda_0 + \alpha^2 t} + \frac{\alpha}{\lambda_0 + \alpha^2 t} W_t. \tag{1.4}$$

Note that the last term on the right hand side of (1.4), behaves like $|W_t|/\alpha t$, which is a typical quantity of learning errors, highly oscillating and there are trajectories where it is slow. Notice finally that taking large values for α speeds up the convergence, and that the characteristic time of concentration (around the point) is multiplied by α while the characteristic time of convergence of the variance is multiplied by α^2.

1.1.2. *Stochastic control.* We now address questions related to the optimization process.

The naive approach. By drawing a randomly on the probability μ_t, we consider the following dynamics of x_t,

$$\mathrm{d}x_t = a\alpha_t \,\mathrm{d}t + \mathrm{d}W_t. \tag{1.5}$$

The link between the equations is explained by the following: In a time interval $(t, t + \mathrm{d}t)$, we draw a according to the law μ_t which gives x_t according to (1.5). On the other hand, x_t provides $\mu_t = \nu_t / \int \nu_t$ through $\mathrm{d}\nu_t(z) = z\, a_t \nu_t \,\mathrm{d}x_t$. We consider, for instance the following control problem in infinite horizon:

$$\mathbb{E}\left[\int \int_0^\infty e^{-\rho t} L(x_t^a, \alpha_t) \,\mathrm{d}\mu_t(a)\right],$$

where $\rho > 0$, and we are looking for a control $\alpha_t = \alpha(x_t)$ which does not depend on a. Now, the two aspects of optimization and learning are coupled. Note that this is not a problem that can easily be formulated in terms of dynamic programming. To write a stochastic control problem, we need to use Girsanov Theorem. One feels like making a dirty version of stochastic control with partial information. This brings us to abandon this naive approach.

Partial information approach. We consider the same process x_t given by the same dynamics (1.5). We assume that α_t is adapted to the following filtration

$$\mathcal{F}_t := \sigma\{x_s \quad | \quad 0 \le s \le t\}.$$

We are interested in the following stochastic control problem:

$$\min_{\alpha_t} J := \min_{\alpha_t} \mathbb{E}\left[\int_0^\infty e^{-\rho t} L(x_t, \alpha_t) \,\mathrm{d}t\right],$$

where $\rho > 0$. We start by making a change of probability through Girsanov's theorem to absorb the drift :

$$J = \mathbb{E}\left[\int_0^\infty e^{-\rho t} L(x_t, \alpha_t) \exp(A \int_0^t \alpha_s \,\mathrm{d}W_s - (A^2/2) \int_0^t \alpha_s^2 \,\mathrm{d}s)\right].$$

Under this new probability,

$$\mathrm{d}x_t = \mathrm{d}W_t,$$

and in addition,

$$J = \mathbb{E}\left[\int_0^\infty e^{-\rho t} L(x_t, \alpha_t) \mathbb{E}\left[e^{A\int_0^t \alpha_s \,\mathrm{d}W_s - (A^2/2)\int_0^t \alpha_s^2 \,\mathrm{d}s} \;\middle|\; \mathcal{F}_t\right]\right].$$

Now we use the idea of Zakaï in this particular case, by introducing a positive measure $f_t(a)$ (where f_0 is a given probability measure) which is defined as follows,

$$\forall \varphi \in C^\infty, \quad \int \varphi(a) f_t(a) := \mathbb{E}\left[\varphi(A) e^{A\int_0^t \alpha_s \, \mathrm{d}W_s - (A^2/2)\int_0^t \alpha_s^2 \, \mathrm{d}s} \;\Big|\; \mathcal{F}_t\right].$$

The expression on the right hand side clearly defines a positive linear form and therefore the existence and uniqueness of f_t is ensured. Now using Zakaï's[1] approach in this particular case, one gets an evolution of f_t according to the following equation

$$\mathrm{d}f_t = a f_t \alpha_t \, \mathrm{d}W_t. \tag{1.6}$$

In particular, we have that

$$f_t = f_0 \exp\left(a\int_0^t \alpha_s \, \mathrm{d}W_s - (a^2/2)\int_0^t \alpha_s^2 \, \mathrm{d}s\right),$$

and the optimal control problem takes the form :

$$\inf_{\alpha \in \mathcal{F}_t} J(x_0, f_0, \alpha) := \inf_{\alpha \in \mathcal{F}_t} \mathbb{E}\left[\int_0^\infty e^{-\rho t} L(x_t, \alpha_t) \int f_t\right], \tag{1.7}$$

where $\alpha \in \mathcal{F}_t$ means that the process α_t is adapted to $\mathcal{F}_t$. Note that (1.7) is a usual Bellman problem, but with variables x and f, where f is a positive measure.

We now give a formal justification of equation (1.6). For any smooth function φ, we have

$$\int \varphi(A) f_{t+h} =$$
$$\mathbb{E}\left[\varphi(A) \exp\left(A\int_0^t \alpha_s \, \mathrm{d}W_s - \frac{A^2}{2}\int_0^t \alpha_s^2 \, \mathrm{d}s\right) \exp\left(A\int_t^{t+h} \alpha_s \, \mathrm{d}W_s - \frac{A^2}{2}\int_t^{t+h} \alpha_s^2 \, \mathrm{d}s\right) \;\Big|\; \mathcal{F}_t\right].$$

Using an asymptotic expansion, we infer that

$$\begin{aligned}\exp\left(A\int_t^{t+h} \alpha_s \, \mathrm{d}W_s - \frac{A^2}{2}\int_t^{t+h} \alpha_s^2 \, \mathrm{d}s\right) &\sim \exp\left(A\alpha_t(W_{t+h} - W_t) - \frac{A^2}{2}\alpha_t^2 h\right) \\ &\sim 1 + A\alpha_t(W_{t+h} - W_t).\end{aligned}$$

The last expansion can be rigorously justified by using Itô's formula instead of asymptotic expansions. It follows that,

$$\begin{aligned}\int \varphi(A) f_{t+h} = \mathbb{E}&\left[\varphi(A) \exp\left(A\int_0^t \alpha_s \, \mathrm{d}W_s - \frac{A^2}{2}\int_0^t \alpha_s^2 \, \mathrm{d}s\right) \Big| \mathcal{F}_{t+h}\right] \\ &+\alpha_t(W_{t+h} - W_t)\mathbb{E}\left[\varphi(A) A \exp\left(A\int_0^t \alpha_s \, \mathrm{d}W_s - \frac{A^2}{2}\int_0^t \alpha_s^2 \, \mathrm{d}s\right) \Big| \mathcal{F}_{t+h}\right].\end{aligned}$$

Using the fact that the information between t and $t+h$ plays no role for the first term in the above expression, we infer that

$$\begin{aligned}\int \varphi(A) f_{t+h} = \mathbb{E}&\left[\varphi(A) \exp\left(A\int_0^t \alpha_s \, \mathrm{d}W_s - \frac{A^2}{2}\int_0^t \alpha_s^2 \, \mathrm{d}s\right) \Big| \mathcal{F}_t\right] \\ &+\alpha_t(W_{t+h} - W_t)\mathbb{E}\left[\varphi(A) A \exp\left(A\int_0^t \alpha_s \, \mathrm{d}W_s - \frac{A^2}{2}\int_0^t \alpha_s^2 \, \mathrm{d}s\right) \Big| \mathcal{F}_t\right],\end{aligned}$$

[1]There is another approach called Kushner's approach, which consists to looking at $f_t / \int f_t$.

so that

$$\int \varphi(A) f_{t+h} = \int \varphi(A) f_t + \alpha_t (W_{t+h} - W_t) \int a \varphi(A) f_t,$$

and finally,

$$f_{t+h} = f_t + \alpha_t (W_{t+h} - W_t) a f_t.$$

For the control problem (1.7), we expect to have an HJB equation which has the following form

$$\rho V + \sup_{\alpha} \left\{ -\frac{1}{2}\Delta V - \frac{1}{2}\frac{\partial^2 V}{\partial f^2}(\alpha a f, \alpha a f) - \left\langle \frac{\partial^2 V}{\partial x \partial f}, \alpha a f \right\rangle - L(x,\alpha) \int f \right\} = 0, \tag{1.8}$$

where the unknown is the function $V(x,f)$ where $x \in \mathbb{R}^d$ and $f \in L^1, L^2 ...$

Reduction to finite dimension. Recall that in the case of a Gaussian, the state of the system is characterized through the functions,

$$\mathrm{d}\lambda_t = \alpha_t^2 \,\mathrm{d}t, \quad \mathrm{d}(\lambda_t A_t) = \alpha_t \,\mathrm{d}W_t, \quad \text{and} \quad \mathrm{d}X_t = \mathrm{d}W_t.$$

More precisely, the state of the system is of dimension $d+2$ (of dimension 3 if $d=1$). Similarly, recall that in the case of discrete distribution $f_0 = \sum_{i=1}^k y_i \delta_{a_i}$, the function $f(t)$ can also be written as $f_t = \sum_{i=1}^k y_i(t) \delta_{a_i}$, and we have a very simple evolution for the y_i, which is

$$\mathrm{d}y_i = a_i \alpha_i y_i \,\mathrm{d}W_t.$$

We thus obtain a reduction in finite dimension: $k+d$.

In general, we prove that the HJB problem (1.7) is a problem of dimension 3 (if $d=1$). In fact, set

$$z_t = \int_0^t \alpha_s \,\mathrm{d}s, \quad \lambda_t = \int_0^t \alpha_s^2 \,\mathrm{d}s$$

and

$$\Psi(z_t, \lambda_t) := \int f_0 \exp\left(a z_t - \frac{a^2}{2} \lambda_t \right) = \int f_t.$$

We start by giving some properties of the function Ψ.

Remark 1.5 (on the function Ψ). Let

$$\Psi(z,\lambda) = \int e^{az - \frac{a^2}{2}\lambda} f_0$$

If f_0 is a probability measure, then $\Psi > 0$, it is decreasing with respect to λ (i.e. $\partial_\lambda \Psi < 0$), convex with respect to a (i.e. $\partial_{zz} \Psi > 0$) and we have the following:

$$\partial_{zz} \Psi = \int a^2 e^{az - \frac{a^2}{2}\lambda} f_0,$$

and

$$\partial_\lambda \Psi = -\frac{1}{2} \int a^2 e^{az - \frac{a^2}{2}\lambda} f_0.$$

We infer in particular that

$$\partial_\lambda \Psi + \frac{1}{2} \partial_{z,z} \Psi = 0.$$

With the above formalism we can completely rewrite our problem of stochastic control in this particular case in the following form,

$$\inf_{\alpha\in\mathcal{F}} \mathbb{E}\left[\int_0^\infty e^{-\rho t} L(x_t,\lambda_t)\Psi(z_t,\lambda_t)\right],$$
$$\mathrm{d}x_t = \mathrm{d}W_t, \quad x_0 = x,$$
$$\mathrm{d}z_t = \alpha_t\,\mathrm{d}W_t, \quad z_0 = z,$$

and

$$\mathrm{d}\lambda_t = \alpha_t^2\,\mathrm{d}t, \quad \lambda_0 = \lambda.$$

A standard stochastic control problem is thus recovered. Note that this formalism generalizes the Gaussian case, and the only advantage in the Gaussian case is that we can explicitly compute the corresponding function Ψ. Now, the equation HJB written in (1.8) is reduced to dimension 3:

$$\rho V + \sup_\alpha \left\{ -\frac{1}{2}\,\partial_{xx}V - \frac{\alpha^2}{2}\,\partial_{zz}V - \alpha\,\partial_{xz}V - \alpha^2\,\partial_\lambda V + L(x,\alpha)\Psi(z,\lambda)\right\} = 0.$$

However, this simplification leads to an additional mathematical difficulty. Let us take an example to illustrate this difficulty.

Example 1.6. If $L \equiv 0$, then $V \equiv 0$ is a solution. However, by taking $V \equiv \Psi$ (and forget about the first part of the equation) we see that this choice cancels the part with "$\sup_\alpha$" in the equation, thanks to the properties of the function Ψ described in Remark 1.5. Thus Ψ is in the kernel of the equation.

We infer that there is basically non-uniqueness for the equation HJB (with reduced dimension). For this purpose, we will normalize the function in the hope to recover the uniqueness. The idea is to perform the following change of the unknown function:

$$V = \Psi(z,\lambda)U.$$

Hence, one gets an equation for U by dividing by $\Psi > 0$:

$$H(U,\Psi) + \rho U - \frac{1}{2}\,\partial_{xx}U = 0,$$

where

$$H(U,\Psi) = \sup_\alpha \left\{ -\frac{\alpha^2}{2}\,\partial_{zz}U - \alpha^2\,\partial_\lambda U - \alpha\,\partial_{xz}U - \alpha^2\frac{\partial_z\Psi}{\Psi}\,\partial_\lambda U - \alpha\frac{\partial_z\Psi}{\Psi}\,\partial_x U - L(x,\alpha)\right\}.$$

Now, the causes of the unbounded character have been reduced. However, a term appeared in two places in the expression above, which is

$$F(\Psi) := \partial_z\left\{\ln(\Psi)\right\},$$

could possibly be the cause of the unbounded character. If we look for $\alpha \in D$ with bounded D, it is possible to apply the general results of the classical theory of HJB equations. Under natural assumptions, we can have a unique solution to this equation provided that $F(\Psi)$ has a linear growth with respect to z. This will depend on f_0, and more

precisely on its confinement. For example, if f_0 has a compact support, the desired property is satisfied. Another interesting example is when $f_0 \sim e^{-\frac{\delta}{2}a^2}$ at infinity. For this example, we have formally (for $\lambda = 0$),

$$\Psi(z,\lambda) = \int e^{-k(a-z/k)^2/2} e^{\frac{z^2}{2k}}.$$

In that case, we have that

$$F(\Psi) = z/k,$$

which is a limit case, to get the linear growth assumption.

Note that we are in situations where we can pay to learn. There are situations where paying for learning is not quite expensive enough. Hence, one can learn immediately through a violent transition, then we get a problem of standard stochastic control. This violent transition is typical of what is called *singular control*.

1.1.3. *Stochastic control with partial information : the general case.* We consider a situation where we observe only a part of a general stochastic system. Consider the following general system

$$\begin{cases} \mathrm{d}X_t = \sigma(X_t, Y_t, \alpha_t)\,(\,\mathrm{d}W_t + b(X_t, Y_t, \alpha_t)\,\mathrm{d}t)\,, \\ \\ \mathrm{d}Y_t = \gamma(X_t, Y_t, \alpha_t)\,(\,\mathrm{d}W_t + b(X_t, Y_t, \alpha_t)\,\mathrm{d}t) + \delta(X_t, Y_t, \alpha_t)\,\mathrm{d}Z_t + \beta(X_t, Y_t, \alpha_t)\,\mathrm{d}t, \end{cases}$$

where W and Z are independent Brownian motions.

Assume that α is adapted to the σ-field generated by X up to time t, which we note

$$\alpha \in \mathcal{F}_X.$$

In addition, we consider the following cost structure

$$J := \inf_{\alpha \in \mathcal{F}_X \text{ and } \alpha \in D} \mathbb{E}\left[\int_0^\infty e^{-\rho t} L(X_t, Y_t, \alpha_t)\,\mathrm{d}t\right].$$

We assume that

$$\sigma \text{ does not depend on } y, \tag{1.9}$$

so that

$$\mathrm{d}X_t = \sigma(X_t, \alpha_t)\,\mathrm{d}W_t + \sigma(X_t, \alpha_t) B(X_t, Y_t, \alpha_t)\,\mathrm{d}t,$$

and we use Girsanov's theorem to get

$$J := \inf_{\alpha \in \mathcal{F}_X} \mathbb{E}\left[\int_0^\infty e^{-\rho t} \int L(X_t, y, \alpha_t) \nu_t(y)\,\mathrm{d}y\,\mathrm{d}t\right],$$

where ν_t is a random measure given by

$$\forall \varphi \in C^\infty, \quad \int \varphi(y)\nu_t(y)\,\mathrm{d}y = \mathbb{E}\left[\varphi(Y_t) \exp\left(\int_0^t b\,\mathrm{d}W_s - \frac{1}{2}\int_0^t b^2\,\mathrm{d}s\right) \Big| \mathcal{F}_t\right].$$

As for the previous cases, we aim to write an equation for ν_t. Using the fact that Z is independent of $\mathcal{F}_t$, we have

$$\begin{aligned}
\mathrm{d}\int \varphi\nu_t = \mathbb{E}\left[L\varphi(Y_t)\exp\left(\int_0^t b\,\mathrm{d}W_s - \frac{1}{2}\int_0^t b^2\,\mathrm{d}s\right)\Big|\mathcal{F}_t\right]\mathrm{d}t \\
+\mathbb{E}\left[b\gamma\,\partial_y\varphi\exp\left(\int_0^t b\,\mathrm{d}W_s - \frac{1}{2}\int_0^t b^2\,\mathrm{d}s\right)\Big|\mathcal{F}_t\right]\mathrm{d}t \\
+\mathbb{E}\left[b\varphi\exp\left(\int_0^t b\,\mathrm{d}W_s - \frac{1}{2}\int_0^t b^2\,\mathrm{d}s\right)\Big|\mathcal{F}_t\right]\mathrm{d}W_t \\
+\mathbb{E}\left[\gamma\,\partial_y\varphi\exp\left(\int_0^t b\,\mathrm{d}W_s - \frac{1}{2}\int_0^t b^2\,\mathrm{d}s\right)\Big|\mathcal{F}_t\right]\mathrm{d}W_t,
\end{aligned}$$

where

$$L := \frac{1}{2}(\gamma^2+\delta^2)\,\partial_{yy} + \beta\,\partial_y.$$

We infer that,

$$\mathrm{d}\int \varphi\nu_t = \left(\int L\varphi\nu_t\right)\mathrm{d}t + \left(\int \gamma\,\partial_y\varphi\nu_t\right)\mathrm{d}W_t + \left(\int b\varphi\nu_t\right)\mathrm{d}W_t + \left(\int b\gamma\nu_t\,\partial_y\varphi\right)\mathrm{d}t,$$

so that

$$\mathrm{d}\nu_t = L^*\nu_t\,\mathrm{d}t + b\nu\,\mathrm{d}W_t - \partial_y(\gamma\nu)\,\mathrm{d}W_t - \partial_y(b\gamma\nu)\,\mathrm{d}t. \tag{1.10}$$

Note that the main difference compared to Zakaï's equation presented in the previous subsection, is that here the coefficients of L depend on the control.

The first two terms in (1.10) are standard. The other terms are less standard and come from correlations. Equation (1.10) is a parabolic equation of evolution, which is conservative. The state of the system (X_t, ν_t) is totally observable, so we get a classical stochastic control problem in the variable (x, ν). As for the particular case explained before, it is possible to write the HJB equation corresponding to this problem. We will not write it here. The aim of this section is to show how to reduce a problem, as general as we can imagine, with partial information, to a problem of stochastic control in complete information with an HJB equation with value function of the form $V(x, \nu)$.

To ensure the wellposedness for all the equations, one needs technical hypotheses (on the coefficients of the SDE), and also some growth hypotheses related to the structure of the problem. Note that assumption (1.9) is structural and not only technical.

Given the results of the previous subsection, one might wonder if it is also possible to reduce the dimension for this general system. It is necessary to have very particular situations so that one can reduce the problem of infinite dimension to a problem of finite dimension. More precisely, in the case where the filtering going from Y_t to ν_t is linear, the dimension can be reduced, and this is the case for the particular example which was previously explained. We shall give a few examples.

Example 1.7. Assume that $\alpha \in D$ (is bounded). Suppose that $L(x, \alpha) \equiv +\infty$ if $a \notin D$ the supremum becomes a maximum on $\alpha \in D$. If we suppose that D is bounded and that L is a continuous Lipschitz function in (x, α). In this situation, there exists a unique viscosity solution V and it is Lipschitz continuous in $\mathbb{R}^3$.

Example 1.8. Assume that $L(x,\alpha) = f(x) + c_0\alpha^2$. The sup will be attained if

$$-\frac{1}{2}\,\partial_{zz}V - \partial_\lambda V - c_0\Psi \leq 0.$$

It seems that this case is degenerate.

Remark 1.9 (about the assumption (1.9)). Consider the general system described above,

$$\mathrm{d}X_t = \sigma(X_t, Y_t, \alpha_t)\,\mathrm{d}W_t + b(X_t, Y_t, \alpha_t)\,\mathrm{d}t,$$

where X_t is observed, α_t is the control and σ, b known, and Y_t is not observed and driven by the following equations

$$\mathrm{d}Y_t = \gamma(X_t, Y_t, \alpha_t)\,\mathrm{d}W_t + \delta(X_t, Y_t, \alpha_t)\,\mathrm{d}Z_t + \beta(X_t, Y_t, \alpha_t)\,\mathrm{d}t,$$

where Z and W are independent, i.e.

$$Z \perp W.$$

We want to come back to hypothesis (1.9) to say that if σ depends on Y, this is not necessarily the end of the story, but that there is a real change. The reason is that when we observe a process, we observe its quadratic variation $< X_t >$. We have that $\mathrm{d} < X_t >= \sigma^2(X_t, Y_t, \alpha_t)$. We infer that this is not only a technical hypothesis, but when it is not satisfied, we get a lot of information about Y.

1.2. **Control of conditioned processes.** We aim in this section to analyze systems that are controlled, and where the decisions we make do not take into account a specific event that could happen on this system. In other words, the observations are not about the system, but about the system conditioned by the fact that we do not take into account a event that could happen. This question is inspired by problems related to biology and especially the interaction between the environment and the survival of species. By comparing experimental results with mathematical models one wonders what is observed? Is it the process, or the process conditioned to the species not being extinct? This question is very general, and arises in many other applications. For example, when we are young, our decisions are generally made knowing that we will not die in the coming years. The majority of the decisions also made by the leaders for example are made knowing that there will not be major problems : stock crash and so on. Another example also in politics is when the candidates for example, condition all their choices to the fact that they will not lose the election. This kind of examples are called "bounded rationality". Compared with conventional control problems, these situations correspond to corrections and it can therefore be considered in some sense a risk measurement. Note that in this type of problems, there is to main cases : easy cases where the conditioning event is static, and difficult cases where the conditioning event is dynamic [2].

In this course, we consider the specific example of a Brownian motion and we use as a condition the fact that the Brownian motion does not touch the boundary fo a given bounded domain, even if we know that the probability that the Brownian touches the boundary converges exponentially fast to 1. When time increases the event is increasingly

[2] depends on time, for example : being or not being alive.

rare, which means that one changes more and more violently the classical problem (the problem without conditioning).

Example 1.10 (exit time for a Brownian motion)**.** Let D a smooth fixed domain of $\mathbb{R}^d$ and $L > 0$. Consider the process X_t, driven by following stochastic differential equation,

$$\mathrm{d}X_t = \mathrm{d}W_t + \alpha_t \,\mathrm{d}t, \quad X_0 = x \in D.$$

We study the control problem associated to the two following criterions.

(1) We use the following optimization criterion :

$$\min_{|\alpha_t| \le L} \mathbb{E}\left(g(X_T) | \tau_x \ge T\right) = \min_{|\alpha_t| \le L} \frac{\mathbb{E}\left(g(X_T 1_{\tau_x \ge T})\right)}{\mathbb{P}(\tau_x \ge T)},$$

where

$$\tau_x = \inf\{s \ge 0 \;\Big|\; X_s \notin \bar{D}\}.$$

Given that we consider a Brownian motion, if D is bounded, we are conditioning with respect to an event that is increasingly rare. Therefore, we expect that for t larger and larger, we deviate more and more from the standard case.

(2) We now consider a criterion with a dynamic conditioning :

$$\min_{\alpha \in L^2_{\omega,t}} \int_0^T \mathbb{E}\left[f(X_t) + \frac{1}{2}\alpha_t^2 \;\Big|\; \tau_x \ge t\right] \mathrm{d}t + \mathbb{E}\left[g(X_T) \;\big|\; \tau_x \ge T\right].$$

Note that, in the case where $D = B(0, R)$, it is expected that when $R \to \infty$ we recover the usual control problems without conditioning.

1.2.1. *Feedbacks.* Consider case (1) in Example 1.10. Note that the functions $u : (t, x) \to \mathbb{E}[g(X_T)1_{\tau_x \ge T}]$ and $v : (t, x) \to \mathbb{E}[1_{\tau_x \ge T}]$ solve parabolic PDE problems of the following type :

$$\partial_t u - \frac{1}{2}\Delta u - \alpha \nabla u = 0, \quad u_{t=0} = g \quad u_{\,\partial D} = 0,$$

and v solves the same equation with $v_{t=0} = 1$. Hence, the minimization problem is of the following form,

$$\min_{|\alpha_t| \le L} u(x, T)/v(x, T).$$

Consequently, we obtain a type of control problem of parabolic equations : one minimizes under a criterion that depends on solutions of two PDE problems. The optimality system for this problem gives rise to four PDEs. By manipulating these PDEs, one realizes that it is much better to look at the conditional law to analyze the problem.

Consider the solution p of the following PDE problem,

$$\partial_t p - \frac{1}{2}\Delta p + \mathrm{div}(\alpha p) = 0, \quad p_{t=0} = p_0, \quad p_{\,\partial D} = 0. \tag{1.11}$$

$$p_0 = \mathcal{L}(X_0) \in L^1.$$

In that case,

$$\mathbb{E}[g(X_T)/1_{\tau \ge T}] = \frac{\int g p(T)}{\int p(T)},$$

so that, the optimization problem (1) in Example 1.10 takes the following form:

$$\min\left\{\left(\int gp(T)\right)\left(\int p(T)\right)^{-1} \quad / \quad p \text{ solves (1.11)}, \quad \|\alpha\|_\infty \le L\right\}. \tag{1.12}$$

We get an optimal control problem of the Fokker-Planck equation, which is in an unusual form. Note that this is a very special case of Mean Field Games problems.

Proposition 1.11. *There exists an optimal control for the problem* (1.11)*. Moreover, for any optimal control α, there exists u which satisfies the following problem :*

$$-\partial_t u - \frac{1}{2}\Delta u + L|\nabla u| = 0, \quad u(T) = g\left(\int p\right)^{-1} - c_0 \quad \text{in } D, \quad u_{\partial D} = 0,$$

where

$$c_0 = \left(\int gp\right) / \left(\int p\right)^2.$$

Moreover, the optimal control is given by $\alpha = -\nabla u/|\nabla u|$ if $\nabla u \neq 0$, otherwise α takes any value.

Note that the MFG coupling is coming through the final condition. Observe that this optimality system depends on on p_0 through p. In particular, if p_0 is a Dirac mass (we know exactly from where we start), then the optimal feedback depends on the starting point. In other words, for each starting point, there is an optimal feedback among the feedbacks.

Proof. The existence of an optimal control is standard: one takes a minimizing sequence α_n which converges weakly to α for the weak * topology of L^∞. Then, if (p_n) is the sequence associated to (α_n), we get the uniform convergence (up to a subsequence) of p_n using the parabolic compactness. Hence, p solves the Fokker Planck problem for α. The only point to be careful about, is to make sure that $\int p_n$ does not tend to 0. So we need lower bounds on $\int p_n$. This is proved using the strong maximum principle, which provides that $p_n \ge \delta > 0$ on all compact subsets of D.

Note that there is no convexity property which allows us to infer that the optimality condition is sufficient. Let $J(\alpha)$ be the cost functional associated to this optimization problem. We minimize on a convex domain, so optimality condition is

$$\left\langle J'(\alpha), \beta - \alpha \right\rangle \ge 0 \qquad \forall |\beta| \le L.$$

The adjoint state will allow us to compute $J'(\alpha)$ as usual. We test the optimality by looking at $\alpha + h\delta\alpha$, where $h \ge 0$ and $\delta\alpha = \beta - \alpha$. Note that δp solves the linearized equation:

$$\partial_t \delta p - \frac{1}{2}\Delta \delta p + \operatorname{div}(\alpha \delta p + (\delta\alpha)p) = 0, \quad \delta p_{t=0} = 0,$$

and the inequality of Euler simply becomes

$$\frac{\int g\delta p(T)}{\int p} - \frac{(\int \delta p)(\int gp)}{(\int p)^2} \ge 0 \qquad \forall |\beta| \le L.$$

To make the dependence on $\beta - \alpha$ explicit in the previous expression, we introduce the adjoint state. We are looking for Γ such that

$$-\partial_t u - \frac{1}{2}\Delta u = \Gamma, \qquad u(T) = \frac{g}{\int p} - \frac{\int gp}{(\int p)^2}.$$

Note that the Euler condition becomes

$$\int u(T)\delta p(T) \geq 0,$$

and on the boundary ∂D, we will assume that $u = 0$. Then Γ is computed as follows:

$$\int_0^T \int_D \int \partial_t(u\delta p) = \int \int \nabla u \,(\alpha\delta p + \delta\alpha p) - \Gamma\delta p.$$

We choose $\Gamma = \alpha\nabla u$ to get rid of the terms in δp. We infer that

$$0 \leq \int \int \nabla u p(\beta - \alpha),$$

so that

$$\int \int \nabla u p\alpha \leq \int \int \nabla u p\beta$$

for any $|\alpha| \leq L$ and $|\beta| \leq L$. Thus

$$\alpha = -L\frac{\nabla u}{|\nabla u|},$$

when $\nabla u \neq 0$, and if $\nabla u = 0$, there is no condition.

□

We now consider the second case of Example 1.10, which is defined by the following optimization criterion:

$$\min_{\alpha \in L^2_{t,x}} \left\{ \int_0^T \frac{\int (f + \frac{1}{2}\alpha^2)p}{\int p}\, \mathrm{d}t + \frac{\int gp(T)}{\int p(T)} \right\}. \tag{1.13}$$

Proposition 1.12. *There exists an optimal control for the problem* (1.13)*. Moreover, for any optimal control α, there exists a function u which satisfies,*

$$\begin{cases} -\partial_t u - \frac{1}{2}\Delta u + \frac{1}{2}\left(\int p\right)|\nabla u|^2 = \frac{f}{\int p} - c(t) \quad \text{in } D, \\ u(T) = \frac{g}{\int p} - \frac{\int gp}{(\int p)^2} \quad \text{in } D, \\ u_{\partial D} = 0, \end{cases}$$

where

$$c(t) = \frac{\int (f + \frac{1}{2}\alpha^2)p}{(\int p)^2},$$

and the optimal control is given by

$$\alpha = -\left(\int p\right)\nabla u.$$

Note that conditioning appears only through $\int p$. When $\int p = 1$, we directly find the cases of classical control. Note also that the above system is a particular MFG system.

Proof. As in the previous case, the existence of an optimal control is related to compactness. However, we must pay attention to a new mathematical difficulty which is related to the fact that α is not necessarily bounded in that case. To deal with this difficulty, we approach the system by a better system where we take in particular α bounded.

Given the previous case, we remove the second term by taking $g \equiv 0$ to simplify the presentation. The optimality conditions are written in this case in the following form:

$$0 = \int_0^T \left\{ \frac{\int (f + \frac{1}{2}\alpha^2)\delta p}{\int p} + \frac{\int \alpha p \delta\alpha}{\int p} - \frac{\int \delta p \int (f + \frac{1}{2}\alpha^2) p}{(\int p)^2} \right\}.$$

Furthermore, we are looking for Γ_1 and Γ_2 such that,

$$0 = \int\int \partial_t u \delta p + u\, \partial_t \delta p = \Gamma_1,$$

$$-\partial_t u - \frac{1}{2}\Delta u = \alpha \nabla u + \Gamma_2, \quad u_{t=T} = 0, \quad u_{\partial D} = 0,$$

and

$$\partial_t \delta p - \frac{1}{2}\Delta \delta p + \operatorname{div}(\alpha \delta p + \delta\alpha p) = 0.$$

By integration by parts, it follows that

$$\Gamma_1 = \int\int (\delta\alpha . \nabla u) p - \Gamma_2 \delta p = 0.$$

By identification, we infer that

$$\alpha = -\left(\int p\right) \nabla u \quad \text{and} \quad \Gamma_2 = \frac{f + \frac{1}{2}\alpha^2}{\int p} - c(t).$$

The claimed result is obtained, by replacing the terms with their values. □

What would also be interesting to do in the context of these models, is to consider the problem as a classic dynamic programming problem plus a correction. This correction corresponds to the risk that we did not consider in classical models, and it would be interesting to quantify this error.

1.2.2. *The limit when $R \to \infty$.* Let α_R, u_R and p_R be respectively the optimal control, the adjoint state and the system state corresponding to $D = B(0, R)$. According to Propositions 1.11 and 1.12, we have in the first case,

$$\alpha_R = -L \frac{\nabla \tilde{u}_R}{|\nabla \tilde{u}_R|},$$

and in the second case

$$\alpha_R = -\left(\int p_R\right) \nabla \tilde{u}_R.$$

Let

$$\tilde{u}_R = u_R + c_R.$$

The constant c_R is used to absorb all the constants in the problem. In fact, for the first example, we have that

$$u_R(T) = \frac{g}{\int p_R} - \frac{\int g p_R}{(\int p_R)^2}.$$

It is therefore quite natural to take

$$c_R := \frac{\int g p_R}{(\int p_R)^2},$$

so that for $t = T$,

$$\tilde{u}_R = \frac{g}{\int p_R}.$$

Note also that $\tilde{u}_R$ satisfies the same equation with the following boundary condition,

$$-\partial_t \tilde{u}_R - \frac{1}{2}\Delta \tilde{u}_R + L|\nabla \tilde{u}_R| = 0, \quad \tilde{u}_R(\partial B_R) = c_R.$$

On the other hand, p_R is a solution to the FP equation which is 0 on the boundary of B_R. Hence for large R,

$$p_R \sim \tilde{p}_R$$

in the L^1 sense, where $\tilde{p}_R$ is the solution on the whole space associated to α_R :

$$\partial_t \tilde{p}_R - \frac{1}{2}\Delta \tilde{p}_R + \operatorname{div}(\alpha_R \tilde{p}_R) = 0, \quad \tilde{p}_R(0) = p_0, \quad \int \tilde{p}_R = 1.$$

In addition, for the other terms, we have a HJB equation with a bundary that tends to infinity and the final condition for $\tilde{u}_R$ tends to g. So at the limit $R \to +\infty$, we know that we recover the equation over the whole space without boundary conditions. This is a standard question for Bellman equations. We deduce by similar arguments that we converge towards the following system of classical dynamic programming:

$$\tilde{u}_R \to u, \quad \tilde{p}_R \to p, \quad \alpha_R \to \alpha = -L\frac{\nabla u}{|\nabla u|}$$

where u satisfies

$$-\partial_t u - \frac{1}{2}\Delta u + L|\nabla u| = 0 \quad \text{in } \mathbb{R}^d, \quad u_{t=T} = g,$$

and

$$\partial_t p - \frac{1}{2}\Delta p + \operatorname{div}(\alpha p) = 0.$$

1.2.3. *The limit $T \to \infty$.* We consider the following problem,

$$\mathrm{d}X_t = \mathrm{d}W_t + \alpha(X_t)\,\mathrm{d}t. \tag{1.14}$$

Note that all what is presented here remains satisfied in more general but non-degenerate frameworks. Set

$$\eta(T) := \mathbb{E}[\phi(X_T) \mid \tau \geq T].$$

We are interested at the limit

$$\lim_{T \to +\infty} \eta(T).$$

If there exists $\pi \in \mathcal{P}(D)$ such that

$$\lim_{T\to+\infty} \eta(T) = \int \phi\pi,$$

and if π does not depend on X_0 (as in ergodic theory), π is called a quasi-stationary measure. Note that it replaces the notion of stationary measures, but for a process conditioned by the probability of leaving D.

Remark 1.13 (Q-processes). We consider the following quantity[3],

$$\eta(t,T) := \mathbb{E}[\phi(X_t) \mid 1_{\tau\geq T}].$$

In the case of bounded domain D, the above quantity converges (up to a subsequence by compactness) to the law of a continuous process called the Q -process. More precisely,

$$\lim_{T'\to+\infty} \mathbb{E}[\phi(X_t) \mid 1_{\tau\geq T'}] = \mathbb{E}[\phi(Q_t)].$$

The process Q is defined on the compact domain and has the following ergodicity property,

$$\lim_{t\to+\infty} \mathbb{E}[\phi(Q_t)] = \int \phi\bar{\pi}.$$

This gives another notion of quasi-stationary measures.

Thanks to the results of previous sections, note that for any $T \geq 0$,

$$\eta(T) := \int \phi p(T) / \int p(T),$$

where p solves the forward FP equation (1.11).

Thanks to Krein-Rutman theorem[4], there exists $\lambda_1 > 0$, and a function $\phi_1 > 0$, which is unique up to a multiplicative constant such that,

$$-\frac{1}{2}\Delta\phi_1 - \alpha.\nabla\phi_1 = \lambda_1\phi_1,$$

and there is a unique π such that, $\pi > 0$, $\int \pi = 1$, and

$$-\frac{1}{2}\Delta\pi + \operatorname{div}(\alpha\pi) = \lambda_1\pi,$$

where π and ϕ_1 are zero at the boundary of the domain. Set

$$A := -\frac{1}{2}\Delta - \alpha.\nabla,$$

and consider the quotient $u(T,x)/v(T,x)$ such that

$$\partial_t u + Au = 0, \quad u_{t=0} = \phi,$$
$$\partial_t v + Av = 0 \quad u_{t=0} = 1,$$

and

$$u = v = 0 \quad \text{on } \partial D.$$

Using Krein-Rutman, we know that

$$u(x,t) \sim e^{-\lambda_1 t}\phi_1(x)c_0,$$

[3]Note the difference with $\eta(T)$.

[4]a generalisation of the Perron-Frobenius theorem to infinite-dimensional Banach spaces.

where c_0 is a constant which depends on initial conditions. To compute c_0, we multiply the first equation by π, and we get that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int u\pi + \lambda_1 \int u\pi = 0,$$

so that

$$\int u\pi = \left(\int u(0)\pi\right) e^{-\lambda_1 t}.$$

We infer, according to the first ansatz that

$$e^{-\lambda_1 t} c_0 \int \phi_1 \pi = \int u\pi.$$

Hence

$$c_0 = \frac{\int u(0)\pi}{\int \phi_1 \pi}.$$

One easily gets analogous results for v by replacing ϕ_1 with 1. Thus

$$\lim_{T\to\infty} \frac{u(T,x)}{v(T,x)} = \frac{\int u_0 \pi}{\int \pi},$$

so that

$$\lim_{T\to\infty} \mathbb{E}[\varphi(X_T) \mid 1_{\tau\geq T}] = \int \phi\pi.$$

We deduce that π is the quasi-stationary measure. Note that this result is related to the compactness obtained by a non-degenerate operator on a bounded domain, and also to the positivity of the operator via the Krein-Rutman Theorem.

Now, we try to understand the Q -measures to see the relationship with the object found above. We have that,

$$\eta(t,T) = \frac{\mathbb{E}[\phi(X_t) 1_{\tau\geq t} \mathbb{E}_{X_t}(1_{\tau>T-t})]}{\mathbb{E}_x(1_{\tau\geq T})}.$$

Thanks to the previous considerations, we know that the right-hand side behaves like

$$\frac{\int u_0\pi}{\int \phi_1\pi} e^{-\lambda_1 t}\phi_1,$$

so that, for t large enough, the previous quantity gives

$$\frac{\mathbb{E}[\phi(X_t) 1_{\tau\geq t}\phi_1(X_t)] e^{-\lambda_1 t}}{\phi_1(x)},$$

by normalizing $\int \phi_1\pi = 1$ and $\int \pi = 1$. We now rely on the limit when $T\to\infty$ to see if there is ergodicity. We have that

$$\lim_{T\to\infty} \frac{\mathbb{E}[\phi(X_t) 1_{\tau\geq t}\phi_1(X_t)] e^{-\lambda_1 t}}{\phi_1(x)} = \int \phi\phi_1\pi.$$

We infer that there exists an invariant measure for the Q-process, and that this invariant law is $\bar{\pi} := \phi_1\pi$. Thus, in these non-degenerate frameworks, we completely identify these notions of quasi-stationary measures by using the first eigen functions. Note that, the two measures (in the two cases $\eta(T)$ and $\eta(t,T)$) are different. This is expected, since in the second case the conditioning is stronger.

If we now take a feedback and consider the first problem. We can expect that the feedback becomes stationary as $T \to +\infty$. If we assume that α is independent of time, then formally we have that

$$\lim_{T\to+\infty} \inf_{\alpha} \mathbb{E}[g(X_T) \mid 1_{\tau\geq T}] = \inf_{\alpha} \int g\pi_\alpha.$$

Formally, we recover an optimization problem at the limit $T \to +\infty$ which has nothing to do with the conventional optimal control problems (without conditioning). This was expected since we are conditioning with respect to events whose probability tends exponentially fast to zero. In this example, we recover an optimal control problem in which the first eigen function is controlled. This type of control problems have their own interest.

In the case where there is no conditioning (framework of the classical control theory), it is known that the control becomes independent of time because (under conditions of ergodicity, and periodic or Neumann boundary conditions) $\lim_{t\to\infty} \nabla u(x,t) = \nabla u_\infty(x)$. This result relies on well known strong uniqueness results. In the case of MFGs, it is more complicated. It is therefore natural to ask the same question for general problems : If we take the problem with $\alpha(x,t)$ which is a feedback on $[0,T]$, does it converge to the stationary problem with $\alpha(x)$?

We introduce a supplement which is a kind of generalization of the notions of stationary and quasi-stationary measures for cases where dependence on time is general.

1.2.4. *Some remarks on stationary and quasi-stationary measures.* All the results of this subsection can be generalized to a general non-degenerate framework. It is well known that the law of a process satisfying a SDE of the form (1.14) with α depending also on t, satisfies the following problem:

$$\partial_t p - \frac{1}{2}\Delta p + \operatorname{div}(\alpha(t,x)p) = 0, \quad p_{t=0} = p_0 \quad \int p = 1.$$

To simplify, we shall take periodic boundary conditions. Now we want to take the limit as $t \to +\infty$ to see what happens. Note that the coefficients are any $\alpha \in L^\infty_{t,x}$. We take a sequence of time (t_n) such that $\lim_n t_n = +\infty$, and

$$\alpha(t_n + t, x) \rightharpoonup \bar{\alpha}(t,x) \in L^\infty(\mathbb{R}_t \times Q_x).$$

We aim at constructing a sliding window, in order to understand if it is possible to construct examples where there is no global stabilization of the measure. We will see that there exists a unique regular $\bar{p} > 0$ such that $\int \bar{p}(t) = 1$, for all $t \in \mathbb{R}$ and which satsfies

$$p(t_n + t, x) \to \bar{p}(t,x),$$

where the convergence above is uniform in x and bounded t. Note that uniqueness insures that $\bar{p}$ inherit all the properties of $\bar{\alpha}$. Namely, if α is constant, then $\bar{p}$ is constant, and if α is periodic, then $\bar{p}$ is periodic, and so on. The function $\bar{p}$ is an object that strongly encodes all the properties of the coefficients that are defined on the whole real line $\mathbb{R}$. The question that now naturally arises is the following : in the case where α depends on t and x, do we have analogues function of first eigen function (relatively to the previous case above)?

There exists a unique couple $(\bar{p}, \bar{\lambda})$ such that $\int \bar{p} = 1$ for all $t \in \mathbb{R}$, and $\int_0^t \bar{\lambda}(s)\,\mathrm{d}s$ is stuck between two linear functions that intersect in 0, such that

$$\partial_t \bar{p} - \frac{1}{2}\Delta \bar{p} + \operatorname{div}(\bar{\alpha}\bar{p}) = \bar{\lambda}(t)\bar{p} \qquad \forall t \in \mathbb{R}, \forall x \in Q \quad \bar{p} > 0,$$

and there exists a unique ϕ_1 such that

$$\partial_t \phi_1 - \frac{1}{2}\Delta \phi_1 - \bar{\alpha}\nabla\phi_1 = \bar{\lambda}\phi_1, \quad \phi_1 > 0, \quad \max \phi_1 = 1, \quad \phi_1(\partial D) = 0.$$

This is a strange notion that, in this case, generalizes perfectly the notion of first proper function. It's not an eigen function, because there is no compactness and we work on the whole real line.

If we now recall the problem of stochastic control of conditioned processes, what we know is that when $T \to +\infty$ the stochastic control problem has the following form:

$$\min_{\alpha \in \mathcal{W}} \left\{ \int g p(x) \right\}, \tag{1.15}$$

where $\mathcal{W}$ is the set of functions $\alpha(t,x) \in L^\infty(\mathbb{R}_t \times D)$ such that there exists $s \in \mathbb{R}$ with $p = p(s,x)$ and p satisfies the problem

$$\partial_t p - \frac{1}{2}\Delta p + \operatorname{div}(\alpha p) = 0, \quad p > 0, \quad \int p = 1, \quad p_{\partial D} = 0.$$

Saying that the feedback becomes stationary, means that the quantity (1.15) remains exactly the same if we take the minimum over functions α depending only on x.

1.2.5. *Generalizations of stationary measures.* We consider the following problem :

$$\partial_t p + A^* p = 0, \quad p_{t=0} = p_0 \geq 0.$$

where

$$A := -\frac{1}{2}\Delta - b.\nabla$$

and assume that $b \in L^\infty$. The notion of stationary measure is clear when we have suitable boundary conditions that make this object exist. In the framework of a bounded domain with Dirichlet conditions, then $\lim_{t\to+\infty} p = 0$ because all the mass ends up leaving from the boundary. On the other hand, whith periodic, or Neumann conditions ($\frac{\partial p}{\partial n} = 0$ on ∂D) and if D is open bounded and regular, the mass is preserved, because all the mass is reflected on the boundary.

We know that in the case where b depends only on x, there exists a unique measure $\bar{p} > 0$ that satisfies $A^*\bar{p} = 0$ and which satisfies the same boundary conditions (implying that $\int \bar{p} = 1$).

Remark 1.14. The last results holds in a more general framework. In fact, the result holds for operators of the form:

$$-\frac{1}{2}\,\partial_{i,j}(a_{i,j}.) - \partial_i(b_i.)$$

with b depending only on x, with possibly jump terms.

What we want to look at now is the case where the coefficient b also depends on t. Let (t_n) be a sequence such that $\lim_n t_n = +\infty$ and consider the sequence $b(t_n + s, x)$, for a given $s \in \mathbb{R}$ and $x \in Q$. Since $b \in L^\infty_{t,x}$, one can always infer that $b(t_n + s, x)$ converges weakly * in L^∞ to $b(s, x) \in L^\infty$. Hence, by compactness one can infer that $p(t_n + t, x)$ converges weakly (up to a subsequence) to a function $p(t, x)$, which satisfies the following problem,

$$\partial_t p + A^* p = 0 \quad x \in Q, \quad t \in \mathbb{R}, \quad \int p(t) = 1.$$

Moreover, using the maximum principle, we have that

$$p \geq 0 \Rightarrow p > 0.$$

Thus, we get a notion which replaces the notion of stationary measure since the limit is a zero of the "eternal" operator. Note also that this object is unique, and fully characterized by the coefficients. The following result collects the above statements.

Theorem 1.15. *If $b \in L^\infty(\mathbb{R}_t \times Q)$, then there exists a unique solution p to the problem*

$$\partial_t p + A^* p = 0, \quad \int p = 1, \quad p \geq 0 \quad \forall t \in \mathbb{R}.$$

Corollary 1.16. *If $b(t_n + ., .)$ converges weakly * in L^∞ to b, then $p(t_n .., .)$ converges weakly to p.*

The following corollary shows that p inherits all the properties related to time of the coefficients.

Corollary 1.17. *i) If $b(t, x) \to b(x)$ then $p(t, x) \to p(x)$ which is the stationary solution of $A^* p = 0$.*

ii) If $b(t, x) \to b$ with b periodic, then $p(t, x) \to p$ which is also periodic.

iii) We have the same result if b is almost periodic and so on.

The last two results are easily deduced by uniqueness, so we only prove the first theorem.

Proof. The proof relies on duality arguments. Existence: we consider the sequence (p_n) defined as the solution of the following problem

$$\partial_t p_n + A^* p_n = 0, \quad t \geq -n, \quad x \in Q, \quad p_{n,t=-n} = p_0,$$

where $\int p_0 = 1$ $\left(\text{for instance } p_0 = \frac{1}{|Q|}\right)$. We get compactness using parabolic estimates, and therefore there exists a sequence $p_{n'}$ which converges to a solution of the problem.

Uniqueness: Consider two solutions p_1, p_2 and set $f \equiv p_1 - p_2$. We have that $\int f = 0$ and $\int |f| \leq 2$. Moreover, f satisfies the following problem,

$$\partial_t f + A^* f = 0.$$

To use a duality argument, we introduce the solution of the following backward problem,

$$-\partial_t u + A^* u = 0 \quad \text{on } [-T, 0] \quad u_{t=0} = u_0 \in C^\infty.$$

Up to a time translation, we aim to prove that $f_{t=0} \equiv 0$. We have that,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int uf = \int Auf - uA^*f = 0.$$

Therefore, we have in particular,

$$\int u_0 f(0) = \int u(T)f(-T).$$

Lemma 1.18. *The oscillation of $u(t)$ is bounded by $c_0 e^{-rt}\,\mathrm{Osc}(u_0)$, i.e*

$$\exists r > 0, \quad \exists c_0 \geq 0, \forall u_0, \quad \forall t \in [-T, 0], \qquad \mathrm{Osc}(u(t)) \leq c_0 e^{-rt}\,\mathrm{Osc}(u_0),$$

where $\mathrm{Osc}(f)$ is the oscillation of the funciton f.

This Lemma is proved by showing that

$$\forall \delta > 0, \exists \theta \in (0,1) \quad \text{ such that } \mathrm{Osc}(u(-\delta)) \leq \theta\,\mathrm{Osc}(u_0).$$

using the strong maximum principal and compactness. Using now Lemma 1.18 we infer that

$$\left|\int u(T)f(-T)\right| \leq \mathrm{Osc}(u(-T))\int |f| \leq 2\,\mathrm{Osc}(u(-T)) \to 0$$

when $t \to +\infty$, and hence we get the claimed result. □

The results presented above are already known in the litterature. Now, we are going to look at a situation that is a bit more tricky and concerns the Q-stationary measures.

1.2.6. *Dirichlet conditions and Q-stationary measures.* Consider a function b which depends only on x for the moment and consider the following problem

$$\partial_t u - \frac{1}{2}\Delta u - b.\nabla u = 0, \quad \text{ in } D, \quad u_{\,\partial D} = 0 \quad u_{t=0} = u_0,$$

where D is an open, regular, and bounded set of $\mathbb{R}^d$. We know that the solution of this problem tends exponentially fast to zero. Moreover, we know by Krein-Rutman Theorem that there exists a unique function $\varphi > 0$ which satisfies $A\varphi = \lambda_1\varphi$ in D, and $\varphi_{\,\partial D} = 0$ with $\lambda_1 > 0$. Note that this result is very general and relies on the fact that the operator A^{-1} is compact in the usual spaces $(L^2, C, ...)$, and on the maximum principle.

If $|u_0| \leq C_0\varphi$, then using the maximum principle we infer that

$$|u(t)| \leq C_0 e^{-\lambda_1 t}\varphi,$$

where φ is the first eigen function. Now we compute the constant C_0 using Krein-Rutman's Theorem for A^* : we know that there exists a unique function $\bar{p}$ such that

$$A^*\bar{p} = \lambda_1\bar{p}, \quad \bar{p} > 0, \quad \text{ and } \bar{p}_{\,\partial D} = 0.$$

Thus,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int u\bar{p} + \lambda_1\int u\bar{p} = 0,$$

so that,

$$\int \left(u(t)e^{\lambda_1 t}\right)\bar{p} = \int u_0\bar{p}.$$

At the limit, we infer that

$$C_0 \int \varphi \bar{p} = \int u_0 \bar{p}.$$

Hence

$$C_0 = \frac{\int u_0 \bar{p}}{\int \varphi \bar{p}}.$$

We deduce that the the good normalization to take is the following

$$\int \bar{p} = 1, \quad \text{and} \quad \int \varphi \bar{p} = 1.$$

If we write the corresponding Kolmogorov equation, we have that

$$\partial_t p - \frac{1}{2}\Delta p + \operatorname{div}(bp) = 0, \quad p_{\partial D} = 0 \quad \text{and } p_{t=0} = p_0.$$

If we interpret the above results, we infer that

$$\lim_{t \to +\infty} p(t) e^{\lambda_1 t} = \frac{\int p_0 \varphi}{\int \bar{p}\varphi} \bar{p},$$

where the convergence is uniform.

We now address the case where b depends also on t. Given what we did before, this situation arises naturally when we write control problems for conditioned processes (b plays to role of the control).

We want to keep the notion of eternal solution. As we did before, consider a sequence (t_n) such that $\lim_n t_n = +\infty$. We know that $b(t_n + t, x)$ converges in $L^\infty_{t,x}$ to a function b as $n \to \infty$ (up to a subsequence). We want to create an eternal solution, however in that case everything vanishes quickly towards 0. Hence, we need to introduce a suitable normalization to keep some information. Let

$$p_n(t, x) = p(t_n + t, x),$$

and consider

$$\tilde{p}_n := \varphi_n(t) p_n(t, x),$$

where φ_n is chosen such that

$$\int \tilde{p}_n = 1.$$

We infer that $\tilde{p}_n$ satisfies the following problem,

$$\partial_t \tilde{p}_n + A\tilde{p}_n = \lambda_n \tilde{p}_n, \quad \tilde{p}_n \geq 0, \quad \int \tilde{p}_n = 1, \quad \forall t \geq 0$$

with

$$\tilde{p}_{n,\partial D} = 0, \quad \text{and} \quad \lambda_n(t) = \varphi_n'(t).$$

We have that

$$\varphi_n = \left(\int p_n \right)^{-1},$$

so that

$$\varphi_n' = -\frac{1}{\left(\int p_n\right)^2} \left(\frac{\mathrm{d}}{\mathrm{d}t} \int p_n \right).$$

So if we write an equation we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int p_n - \frac{1}{2}\int_{\partial D}\frac{\partial p_n}{\partial \nu}\,\mathrm{d}S = 0.$$

Since p is positive and satisfies $p_{\,\partial D} = 0$, then according to Hopf's maximum principle, we have that

$$\frac{1}{2}\int_{\partial D}\frac{\partial p_n}{\partial \nu}\,\mathrm{d}S < 0,$$

and so

$$\varphi_n' > 0.$$

This allows us to understand why (λ_n) remains bounded. We write λ_n as a function of $\tilde{p}_n$. In fact, we have (with the same computation as above)

$$\lambda_n = \int_{\partial D}\left(-\frac{\partial \tilde{p}_n}{\partial \nu}\right)\,\mathrm{d}S.$$

We infer formally, that at the limit $n \to \infty$, we get the following problem,

$$\partial_t p + A^* p = \lambda p, \quad (t,x) \in \mathbb{R}\times D$$

and

$$p_{\,\partial D} = 0, \quad \lambda \geq 0, \quad \text{and } \int p = 1, \quad \forall t \in \mathbb{R}.$$

We proved formally that we obtain the equivalent of stationary measures, that we have studied in the classical case, except that now the eigenvalues are different for different function b, and therefore the λ is not necessarily constant.

In the following result, we prove uniqueness of p and λ, and get a kind of generalization of the notion of eigenvalue.

Theorem 1.19. *Let $b \in L^\infty(\mathbb{R}_t, \mathbb{R}_x)$, then there exists a unique pair (p,λ), such that*

$$\partial_t p + A^* p = \lambda p, \quad x \in D, \quad t \in R,$$

$$p \geq 0, \quad \text{in } D, \quad \text{and} \quad \int p(t) = 1, \quad \forall t \in \mathbb{R}.$$

Applications of Theorem 1.19. We recall in the case of feedbacks stochastic control problems that we have found in the two cases of Example 1.10 the following results:

i) In the first case we have the following optimization problem,

$$E_1 := \min\left\{\frac{\int p(T)g}{\int p(T)} \;/\; |\alpha(t,x)| \leq L,\ \partial_t p - \frac{1}{2}\Delta p + \mathrm{div}(\alpha p) = 0,\ p_{\,\partial D} = 0,\ p_{t=0} = p_0\right\}.$$

ii) In the second case, we have the following optimization problem,

$$E_2 := \min\left\{\frac{\int (\frac{1}{2}|\alpha|^2 + f)p(t)}{\int p(t)}\,\mathrm{d}t \;/\; \partial_t p - \frac{1}{2}\Delta p + \mathrm{div}(\alpha p) = 0,\ p_{\,\partial D} = 0,\ p_{t=0} = p_0\right\}.$$

The question we now ask is: what is the limit problem as $T \to \infty$? Consider the example (i). If the optimal control becomes independent of time, then the limit problem is the following,

$$\min \left\{ \int p_\infty g \ / \ |\alpha(x)| \leq L, \ -\frac{1}{2}\Delta p_\infty + \operatorname{div}(\alpha p_\infty) = \lambda p_\infty, \ \lambda \in \mathbb{R}, \ \int p = 1, \ p_{\partial D} = 0 \right\}.$$

This is a well known MFG problem which has a unique solution. Note that, we started from a problem which doesn't have necessarily a unique solution. On the other hand, for the example (ii), we can expect that $\frac{1}{T}E_2(T)$ converges to the following expression

$$\min \left\{ \int \left(\frac{1}{2}|\alpha|^2 + f \right) p_\infty \quad / \quad -\frac{1}{2}\Delta p_\infty + \operatorname{div}(\alpha p_\infty) = \lambda p_\infty, \quad \int p_\infty = 1, \quad p_{\infty,\partial D} = 0 \right\}.$$

Using the results of the previous sections, one checks that the optimality condition in that case is given by,

$$\alpha = -\nabla u,$$

where u is defined by the following problem,

$$-\frac{1}{2}\Delta u + \frac{1}{2}|\nabla u|^2 = \lambda u + f - c, \quad \int u = 0,$$

$$-\frac{1}{2}\Delta p_\infty + \operatorname{div}(\alpha p_\infty) = \lambda p_\infty, \quad \int p_\infty = 1, \quad p_\infty \geq 0,$$

and

$$u = p_\infty = 0 \quad \text{ on } \partial D,$$

where $c \in \mathbb{R}$. The above problem is a stationary MFG system. One can show that there exists a unique solution to this problem, at least for small f. Note also that in case (ii), the functional of the limit problem is convex in α. Thus obviously the optimization problem has a unique solution. This is not the case for the original functional (before taking the limit $T \to +\infty$).

Consider now the example (i) at the limit $T \to +\infty$, without the assumption that α becomes independent of t. One checks that the first optimization problem converges to the following optimization problem

$$\inf_{\alpha,p} \int p(0) g,$$

where the infimum is taken over $|\alpha(t,x)| \leq L$, and p satisfying

$$\partial_t p - \frac{1}{2}\Delta p + \operatorname{div}(\alpha p) = \lambda p, \quad p \geq 0, \quad p_{\partial D} = 0, \quad \int p = 1, \quad \lambda \in \mathbb{R}.$$

To show this result, it is enough to put the origin of time in T. The control α remains arbitrary and is defined for any time. On the other hand, the function p is defined for all time, and normalized. Hence, it converges to the unique solution of the theorem. We deduce that we recover the same problem, except that in place of the stationary measure, we will have the value at 0 of the generalization of stationary measures that we already pointed out.

Now we want to know if the minimum of the problem above is independent of time, which will show that the above problem is the right problem. To prove that the minimum is independent of time, it is enough to check that the minimum is unique. In fact, if there

is a unique minimum, the invariance by time translation proves that the minimum must be independent of time. In other words, if we have uniqueness (convex problem) then $\alpha_{opti}(. + s)$ is still optimal, which implies that α_{opti} is independent of time. We now give the uniqueness for Theorem 1.19.

Proof of Theorem 1.19. Consider two solutions (p, λ) and (q, μ). We shall first reduce to the case $\mu = \lambda$, by constructing a solution which is not normalized, but which has the same time depending eigenvalue. Then we conclude that the new p and the q transformed are equal. In other words, using a suitable transformation we consider a solution $(\tilde{q}, \lambda)$ (in this case we may have $\int \tilde{q} \neq 1$). Then, prove that $p \equiv \tilde{q}$ and thus $\int p = 1 = \int \tilde{q}$, which gives the claimed result. We start with the following observation: if we have a solution (p, λ), we can find an eternal solution (u, λ) for the following dual problem

$$-\partial_t u + Au = \lambda u, \quad u > 0, \quad u_{\partial D} = 0,$$

such that

$$\int up = 1, \quad \forall t \in \mathbb{R}.$$

This result is not difficult to have because for this λ we have that,

$$\frac{\mathrm{d}}{\mathrm{d}t} \int up = 0. \tag{1.16}$$

In fact, identity (1.16) allows us to prove some bounds. To get the existence of an eternal solution, we solve the problem on $(-\infty, T)$, $T > 0$ and we take the limit as $T \to +\infty$. Hence, identity (1.16) is almost an L^1 bound, and we conclude the proof using duality and approximation techniques. On other hand, taking the same u we have that

$$\frac{\mathrm{d}}{\mathrm{d}t} \int uq = (\mu - \lambda) \int uq,$$

so that

$$\int u(t)q(t) = \exp\left(\int_s^t (\mu(s) - \lambda(s))\,\mathrm{d}s\right) \int u(s)q(s).$$

Since the function $w \to \int u(w)q(w)$ is uniformly bounded, we infer that $\int_s^t (\mu(s) - \lambda(s))\,\mathrm{d}s$ is bounded independently of s and t. Hence, if

$$\tilde{q}(t) = q(t) \exp\left(\int_0^t (\lambda(s) - \mu(s))\,\mathrm{d}s\right),$$

we may have $\int \tilde{q} \neq 1$, however $\tilde{q}$ satisfies the following problem,

$$\partial_t \tilde{q} + A\tilde{q} = \lambda \tilde{q},$$

and moreover $\tilde{q}$ is bounded for all $t \geq 0$. Now we will use an argument of L.Tartar. The idea is to introduce the following object

$$f = \sqrt{p\tilde{q}}.$$

If now we look at the equation satisfied by f, we have that

$$\begin{aligned}\partial_t f + Af - \lambda f &= \frac{1}{2}(\partial_t p + Ap)\sqrt{\frac{\tilde{q}}{p}} - \lambda f + \frac{1}{2}(\partial_t \tilde{q} + A\tilde{q})\sqrt{\frac{p}{\tilde{q}}} + \frac{1}{2}|\nabla p|^2 \frac{\sqrt{\tilde{q}}}{(\sqrt{p})^3} \\ &\quad + \frac{1}{2}|\nabla \tilde{q}|^2 \frac{\sqrt{p}}{(\sqrt{\tilde{q}})^3} - \nabla p.\nabla \tilde{q}.\frac{1}{\sqrt{p\tilde{q}}}\end{aligned}$$

By concavity, we can easily deduce the sign of the terms on the right hand side. After simplification we have that

$$\partial_t f + Af = \lambda f + \Gamma$$

where

$$\Gamma = 2\sqrt{p\tilde{q}}|\nabla\sqrt{p} - \nabla\sqrt{\tilde{q}}|^2 \geq 0$$

is the dissipation term. The next step is to notice that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int uf = \int u\Gamma,$$

so that

$$\int_{-\infty}^{+\infty}\int u\Gamma < \infty.$$

Taking into account the sign of the function, we infer that

$$\lim_{t\to-\infty}\int u\Gamma = 0,$$

which implies that Γ tends to 0 and consequently

$$\lim_{t\to-\infty}\nabla\sqrt{p} - \nabla\sqrt{\tilde{q}} = 0$$

This is how we get an information that tells us that p and $\tilde{q}$ are close as $t \to -\infty$. Since the two functions are zero on the boundary, we infer that

$$\lim_{t\to-\infty}\|(p-\tilde{q})(t)\|_{L^1} = 0.$$

Then, thanks to the maximum principle and the normalization of the mass we get that

$$\lim_{t\to-\infty}\sup_{s\geq -t}\|p(s) - \tilde{q}(s)\|_{L^1} = 0,$$

and so

$$p(s) \equiv \tilde{q}(s), \quad \forall s \in \mathbb{R}.$$

□

1.2.7. *The control of processes conditioned by an event A.* Note that the conditioning considered in the previous subsections is a particularly complicated case. In fact, in the previous cases, the conditioning event depends on the entire trajectory, changes with time, and is a related to the control. We now consider a case where the process is conditioned with respect to a given event A.

Consider the following dynamics,

$$\mathrm{d}X_t = \alpha_t\,\mathrm{d}t + \mathrm{d}W_t,$$

where W_t is a Brownian motion and α_t is a control adapted to the σ-field generated by (W_t). We consider the following cost function,

$$J := \mathbb{E}\left[\int_0^T \left(\frac{1}{2}|\alpha_t|^2 + f(x_t)\right)\,\mathrm{d}t \;\middle|\; A\right] = \frac{\mathbb{E}\left[\left(\int_0^T \left(\frac{1}{2}|\alpha_t|^2 + f(x_t)\right)\,\mathrm{d}t\right) 1_A\right]}{\mathbb{P}(A)}.$$

A trivial case is when the event A is independent of W and X. In this case, the conditioning does not affect the control problem. Mathematically, the cost function simply becomes

$$J = \mathbb{E}\left[\int_0^T \left(\frac{1}{2}|\alpha_t|^2 + f(x_t)\right)\,\mathrm{d}t\right].$$

Hence, the interesting case is when we have correlations. Let us consider the following special case

$$1_A = 1_{A_0}(Y_T),$$

where A_0 is a given set, and Y_t is another stochastic process which satisfies a stochastic equation of the form

$$\mathrm{d}Y_t = (...)\,\mathrm{d}B_t, \tag{1.17}$$

so that correlations can be expressed through the dynamic. The dynamics (1.17) could be complex, but B_t is a Brownian motion. In order the avoid the trivial case, we assume that B_t and W_t have the following correlation

$$[\,\mathrm{d}W_t,\,\mathrm{d}B_t] = \rho\,\mathrm{d}t.$$

Note that a simpler case is when $\mathbb{P}(A)$ does not depend on the control. The most important quantity to understand in that case is the following,

$$\inf_{\alpha_t} \mathbb{E}\left[\left(\int_0^T \left(\frac{1}{2}|\alpha_t|^2 + f(x_t)\right)\,\mathrm{d}t\right) 1_{A_0}(B_T)\right].$$

This problem can be reduced to a standard stochastic control problem with more variables. In fact, we can write (at least in law),

$$B_t = \rho W_t + \sqrt{1-\rho^2}Z_t, \quad Z \perp W.$$

In this case we can reduce the size of the problem by considering the following function,

$$\phi(x) := \mathbb{E}_z\left[1_A\left(\rho x + \sqrt{1-\rho^2}Z_T\right)\right].$$

Now we can handle this case exactly as the exit times problems, introducing conditioning laws.

We consider the following simple example to simplify the presentation,

$$\inf J := \inf_{|\alpha_t|\le L} \mathbb{E}\left[g(X_T) \;\middle|\; A\right].$$

The quantity we are interested in is

$$\frac{\mathbb{E}\left[g(X_T)1_{A_0}(Y_T)\right]}{\mathbb{E}[1_{A_0(Y_T)}]},$$

and there is a coupled system of the form

$$(\mathrm{d}X_t, \mathrm{d}Y_t) = (.....)(\mathrm{d}W_t, \mathrm{d}B_t) + (...)\,\mathrm{d}t, \quad [\mathrm{d}W_t, \mathrm{d}B_t] = \rho\,\mathrm{d}t.$$

In this case there may be coupling, and strong correlations with α, so that we can be very close to the situation of exit times (where the coupling is very strong). As for the case of exit times, we can interpret this by the joint law of (x, y). We thus find the same equations, except that we have no boundary conditions since there is no exit time. We get instead an equation on the whole space (or with periodic conditions). Hence, if we consider α of feedback type, the minimization problem takes the following form:

$$\inf_{|\alpha|\leq L} \frac{\int\int q(x,y,T)g(x)1_{A_0}(y)\,\mathrm{d}x\,\mathrm{d}y}{\int\int q(x,y,T)1_{A_0}(y)\,\mathrm{d}x\,\mathrm{d}y}.$$

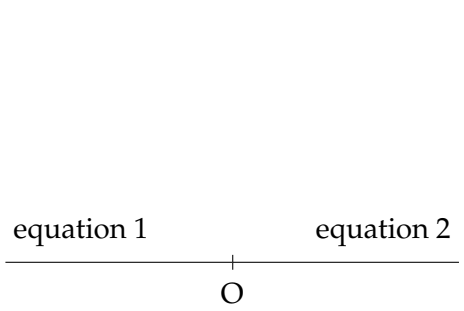
Moreover, if L is the generator of the process (X, Y), then q satisfies

$$\partial_t q + L^* q = 0, \quad q_{t=0} = q_0 \text{ (the initial law)}.$$

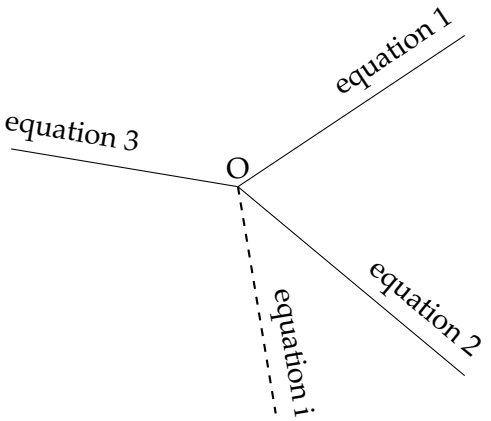
Again, we find a control problem for the Fokker-Planck equation under the above constraints, which is exactly what we found in the case of exit times. We infer that the mathematical formalism introduced for the conditioning problems with exit time is robust.

2. Interfaces and junctions

Interface problems are problems where we have two different equations in two distinct regions of space. Junction problems are situations where we have different branches that join at a point, and where we have an equation on each branch. In general, this kind of problems are idealizations of reality, and from a mathematical point of view, they are limits of problems which are well posed in natural spaces.



Interface problem

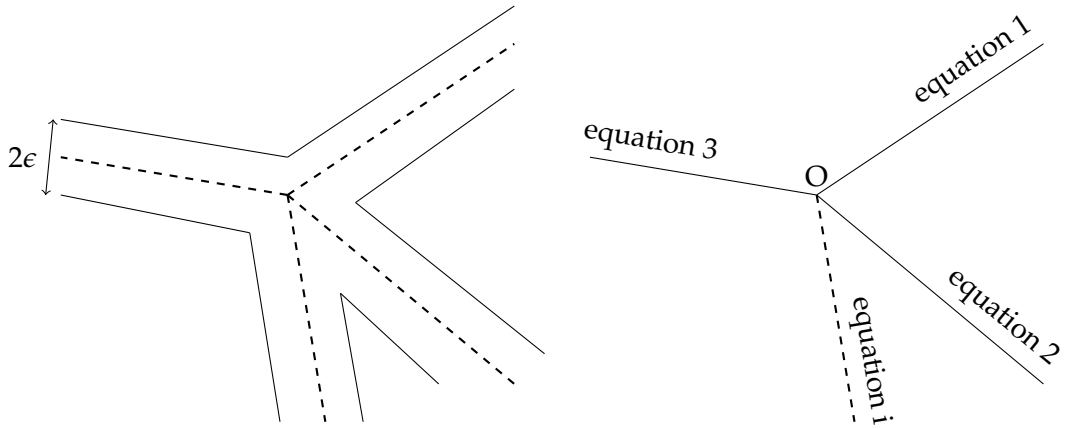


Junction problem

The above drawings illustrate interface problems and junction problems in the one dimensional case. These situations are generalized to spaces of superior dimension considering plans or hyper surfaces. The junction point becomes a junction line and so on. This kind of problems typically appear in the study of networks, traffic flow and also in economy.

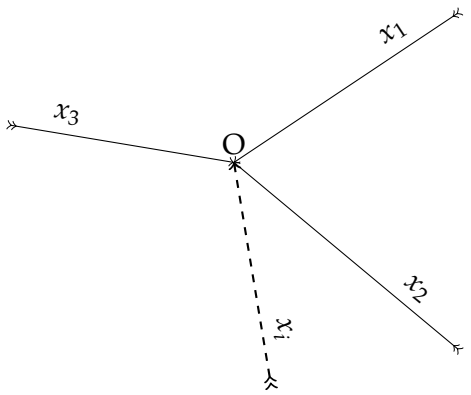
To illustrate the fact that junction problems are limits of well posed problems, we consider an example from electrostatics on electrical wires. Writing the laws governing the system, we find equations with Laplacien operator on a domain of $\mathbb{R}^3$ and which has as a parameter the thickness ϵ of the wire. The junction problem is obtained by taking the limit when ϵ tends to 0. One recovers in particular Kirchhoff law at the junction point under the hypothesis that all the wires have the same diameter. Without this hypothesis we find an analogous law, obtained by multiplying the currents by the related weight of the diameters.

All the equation considered in this section are stationary. In the time dependent case, we should also pay attention to the evolution of the interface in time: if the interface surface is deformed with time or not. We want to develop methods that work in the same way when the deformation of the interface in time is known, and the behaviors will be different according to the equations we have. One can have equations where the analysis is based on the maximum principle, and therefore mathematically, on viscosity solutions. One can also have conservative equations and nonlinear conservation laws. However, we didn't studied yet the case of propagation phenomenons. But, we believe that this case can be managed if we manage correctly the associated elliptic operator.



The initial problem (for instance in $\mathbb{R}^3$) *The limit problem $\epsilon \to 0$*

2.1. **Nonlinear elliptic systems of second order.** We start with the case of fully nonlinear elliptic systems of second order, and we will look at junction situations, where the junction point is 0. We consider the following junction problem,



with i branches, and we consider the following equation on each branch,

$$F_i(x_i, \partial_{x_i} u, \partial^2_{x_i} u) + u = f(x_i), \quad x_i < 0, \tag{2.1}$$

where $F_i(x, p, z)$ and f are regular functions and we have for all (x, p, z),

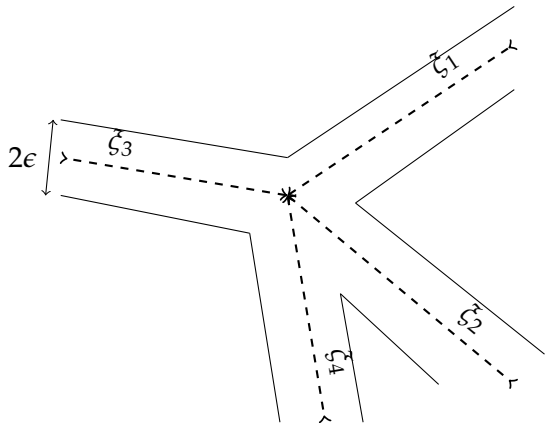
$$-\mu \leq \partial_z F^i \leq -\nu < 0, \quad \mu, \nu > 0. \tag{2.2}$$

The natural question that arises is the following: what is the condition that we must chose in O in order to have a well posed problem on all the junction?

For a given $\{u_i(0)\}_{1 \leq i \leq k}$, the problem is decoupled and we get a unique solution in $C^{2,1}$ on each branch. In other words, the set of solutions is parameterized by the k variables $\{u_i(0)\}_{1 \leq i \leq k}$. We will restrict ourselves to the following situation:

$$u_1(0) = ... = u_k(0).$$

To fix the ideas, we start by looking at an electrostatic problem. Consider $k = 4$ electric wires with a junction in O. We start by considering the problem in $\mathbb{R}^3$:



The domain Σ_ϵ

We aim to recover Kirchhoff's law. Writing the laws of electrostatics, we find equations of the following type,

$$-\Delta u + u = \rho, \quad \text{in } \Sigma_\epsilon, \tag{2.3}$$

where Σ_ϵ is the domain described by the drawing. Note that the equation written above is itself an approximation because in general we can also have a spatial modulation, which is very important if the wires do not have the same electrical properties. The natural boundary condition to consider on $\partial\Sigma_\epsilon$ is the following :

$$\frac{\partial u}{\partial n} = 0 \quad \text{in} \quad \partial\Sigma_\epsilon.$$

Note that the term of order zero in (2.3) is added to avoid additional conditions at infinity, which is in a normal situation a known value of the potential. It is expected that at the limit $\epsilon \to 0$, one recovers four branches, and that on each branch, one has an equation of the following form,

$$-\partial^2_{x_i} u^i + u^i = \rho, \quad x_i < 0.$$

Note that the variable x_i in the equations above, is the projection of vectors of Σ_ϵ on the directions $\xi_i \in \mathbb{R}^3$. In other words, the variable of the limit equation is $x_i\xi_i$. However at this stage, it's not clear why the transverse derivative of the Laplacian disappears. As we will see later, this property is related to the nature of the boundary conditions on $\partial\Sigma_\epsilon$. To recover Kirchhoff's relation, we should prove that at the limit $\epsilon \to 0$ we have that

$$\sum_{i=1}^{k} \frac{\partial u_i}{\partial x_i}(0) = 0, \quad u_1(0) = ... = u_k(0).$$

More generally, if we consider that the wires have different diameters, one should recover a relation of the form,

$$\sum_{i=1}^{k} \lambda_i \frac{\partial u_i}{\partial x_i}(0) = 0, \quad u_1(0) = ... = u_k(0), \quad \lambda_i > 0.$$

This general relation can be obtained at least formally, by writing the law of conservation of currents over a small volume.

Now, what happens when we have a nonlinear equation ? We can no longer have a formal proof using the law of conservation of currents, because we can no longer integrate by parts. Do we still have Kirchhoff relations ?

A priori, we expect to have a nonlinear relation, and more precisely a nonlinear relation of the different derivatives. Namely a relation on the following form

$$G(\partial_{x_1} u, ..., \partial_{x_k} u, u)(0) = 0, \tag{2.4}$$

where G is a non linear function, and if we note the variables of G as $G(p_1, ..., p_k, p_0)$ we have that

$$0 < \nu \leq \partial_{p_i} G \leq \mu. \tag{2.5}$$

As we will see, the function G depends on the equations but also on the geometry, and is globally determined by a sort of a cell problem. To understand the behavior of this system, it is necessary to analyze the transition between the different tubes by zooming.

Proposition 2.1. *There exists a unique solution $u \in C^1$ to equation* (2.1) *with the conditions* (2.2)*,* (2.4) *and* (2.5)*.*

Remark 2.2. We can get bounds easily on u in $C^{1,1}$ by the maximum principle since we are in dimension 1. This makes it possible to have bounds on $\partial^2_{x_i} u$. The regularity of a completely nonlinear equation in one dimension is simple. In higher dimensions the estimates are much more complicated.

Proof. The proof is standard. Existence and regularity is obtained following the ideas of the previous Remark. We recall the proof of uniqueness which works also for higher dimension. Let u and v be two solutions. We note each branch by Σ_i (the closed branch). We proceed by contradiction and assume that $\sup(u - v) > 0$ (for instance), then using the maximum principle we have

$$\max_{\Sigma_i}(u - v) = (u - v)(0) > 0.$$

As the maximum is reached in 0 then

$$\partial_{x_i} u(0) \geq \partial_{x_i} v(0).$$

Moreover, since

$$G(\partial_{x_1} u, ..., \partial_{x_k} u, u)(0) = G(\partial_{x_1} v, ..., \partial_{x_k} v, v)(0) = 0,$$

we infer by the growth of G that for all i,

$$\partial_{x_i} u(0) = \partial_{x_i} v(0).$$

Going to the second order condition we get

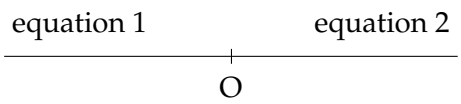
$$\partial^2_{x_i} u(0) \leq \partial^2_{x_i} v(0).$$

Now using the equation for the u_i, and the fact that F is decreasing, we deduce that

$$u(0) \leq v(0),$$

which proves the claimed result. □

2.1.1. *Determination of the effective G.* Let us consider the very simple case in $\mathbb{R}$ given by the following drawing :

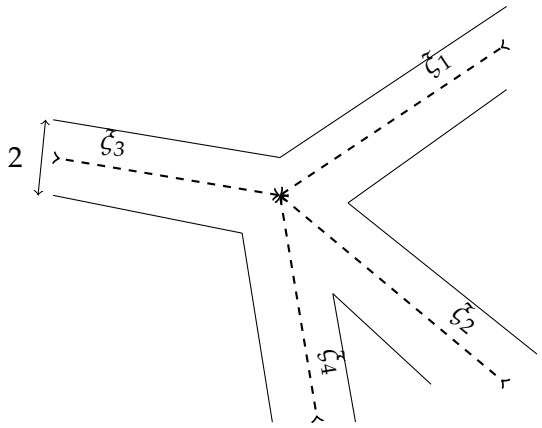


In that case Kirchhoff conditions arises naturally by regularizing. In fact, using a regularization u_ϵ of u in order to have $u_\epsilon \in C^1(\mathbb{R})$ we naturally have

$$\partial_x u_\epsilon(0_-) = \partial_x u_\epsilon(0_+), \quad u_\epsilon(0_-) = u_\epsilon(0_+),$$

which are Kirchhoff conditions in this particular case.

In general, we use the multi-dimensional approximation for the determination of the effective G. As for the previous example from electrostatics, we consider the following domain:



The domain Σ_1

To generalize, we consider the domain Σ_1 in $\mathbb{R}^d$. The sections are balls of diameter 2, and between the tubes a junction is formed, the shape of which does not matter. For any $\epsilon > 0$ set

$$\Sigma_\epsilon = \epsilon\Sigma_1,$$

and consider the following equation in Σ_ϵ

$$F(x, D^2u_\epsilon) + u_\epsilon = 0 \quad \text{in } \Sigma_\epsilon.$$

We suppose that F is Lipschitz continuous, $F(x, 0)$ bounded, F uniformly elliptic and

$$-\mu I \leq \frac{\partial F(x, A)}{\partial A}(x, A) \leq -\nu I,$$

where $\nu > 0$. We consider the following natural boundary conditions,

$$\frac{\partial u_\epsilon}{\partial n} = 0 \quad \text{on } \partial\Sigma_\epsilon.$$

An extension of the above system that is quite natural, is to consider the following equation on Σ_ϵ

$$F\left(\frac{x}{\epsilon}, D^2u_\epsilon\right) + u_\epsilon = 0. \tag{2.6}$$

We now give an Antsatz to explain the ideas of the construction. One start by guessing a candidate solution $\bar{u}$. Then, one construct an approximate solution of the problem in ϵ.

Let $\bar{u}$ be a candidate for the limit solution. We consider the following quantity by adding a small correction to $\bar{u}$ in order to zoom in the junction region,

$$w(x) := \bar{u}(x) + \epsilon v\left(\frac{x}{\epsilon}\right) \quad \forall x \in \Sigma_\epsilon.$$

The function v is used to make the correction in scale 1, and to compress on the scale ϵ. It must have the right behavior in order to be able to extend the derivatives to infinity. More precisely, if x_i is the coordinate with respect to the direction ξ_i and y is the transverse coordinate we must have

$$\lim_{x_i \to +\infty} v(x_i, y)/x_i = p_i,$$

where

$$p_i := -\partial_{x_i} \bar{u}(0).$$

We now build the good v in order to have

$$\lim_{\epsilon \to 0} \|u_\epsilon - (\bar{u} + \epsilon v)\|_\infty = 0.$$

Hence, we naturally expect that there must be some conditions on p_i is order to have the existence of such an object v. In other words, the condition at the limit (of type $G(p_1, ..., p_k)(0) = 0$) is a condition on the fact that one can find such a function v. Let us find an equation for v. If we replace u_ϵ by $\bar{u} + \epsilon v$ in equation (2.6), we have that

$$\epsilon F\left(\frac{x}{\epsilon}, D^2\bar{u}(x) + \frac{1}{\epsilon} D^2 v\left(\frac{x}{\epsilon}\right)\right) + \epsilon \bar{u}(x) + \epsilon^2 v\left(\frac{x}{\epsilon}\right) = 0.$$

If we think of a linear operator of type Δ, we see that at the order zero, we must have an equation on Σ_1 of the form

$$\epsilon F\left(y, \frac{1}{\epsilon} D^2 v(y)\right) \simeq 0.$$

Thus, we need to make the following assumption,

$$\lim_{\epsilon \to 0} \epsilon F\left(y, \frac{1}{\epsilon} A\right) = \bar{F}(y, A). \tag{2.7}$$

Note that with this definition, $\bar{F}$ satisfies

$$-\mu I \leq \frac{\partial \bar{F}(x, A)}{\partial A}(x, A) \leq -\nu I, \quad \text{and} \quad \bar{F}(y, 0) = 0.$$

Example 2.3. The assumption (2.7) is satisfied for the following general class of non linearity:

$$\inf_\alpha \sup_\beta \left(-a_{i,j}^{\alpha\beta} \partial_{i,j} u - f^{\alpha\beta}(x)\right)$$

where

$$\nu I \leq a_{i,j}^{\alpha\beta} \leq \mu I.$$

If we put $y = \frac{x}{\epsilon}$ we have that

$$\lim_{\epsilon \to 0} \inf_\alpha \sup_\beta \left(-a_{i,j}^{\alpha\beta}(y = \frac{x}{\epsilon})\, \partial_{i,j} u - \epsilon f^{\alpha\beta}(x)\right) = \inf_\alpha \sup_\beta \left(-a_{i,j}^{\alpha\beta}(y)\, \partial_{i,j} u\right).$$

We thus obtain the limit $\bar{F}$ for this class of nonlinearity.

With assumption (2.7), we infer that we need to find a v that solves

$$\bar{F}(y, D^2 v) = 0, \quad \text{in } \Sigma_1,$$

such that

$$\lim_{x_i \to +\infty} \frac{v(x_i, y)}{x_i} = p_i,$$

and we add also Neumann boundary condition on $\partial\Sigma_1$,

$$\frac{\partial v}{\partial n} = 0 \quad \text{in } \partial\Sigma_1.$$

The existence and uniqueness of such an object[5] is a compatibility condition between $(p_i)_{1 \le i \le k}$. Since we have made the necessary scale changes, the condiiton

$$G(p_1, ..., p_k) = 0$$

must be the necessary and sufficient condition for the existence of such a v.

We now specify what we must choose as a function F^i for the boundary problem in order to ensure convergence. Let x_i be the coordinate with respect to the direction ξ_i. For $|x_i|$ big enough (far from 0), we have the following equation

$$F_i(x_i, \partial^2_{x_i} \bar{u}) + \bar{u} = 0.$$

The only error is with respect to the transverse variable which is of order ϵ. The equation that one gets is the following

$$F_i(x_i, z) = F_i^\epsilon(x_i \xi_i, z \xi_i \otimes \xi_i).$$

Theorem 2.4. *There exists a unique constant $c(p_1, ..., p_k)$ (up to a transformation) which satisfies the following: there exists a unique solution $v \in C^1$ (up to an additive constant) to the following problem*

$$\bar{F}(y, D^2 v) = 0 \quad \text{on } \Sigma_1, \quad \text{and} \quad \frac{\partial v}{\partial n} = 0 \quad \text{on} \quad \partial\Sigma_1,$$

such that

$$\lim_{x_i \to \infty} v(x_i, y_i)/x_i = p_i,$$

uniformly in y_i, if and only if

$$c(p_1, ..., p_k) = 0.$$

Moreover c is strictly increasing with respect to each of the variables and Lipschitz continuous. On the other hand, there exists a unique function ϕ strictly decreasing such that

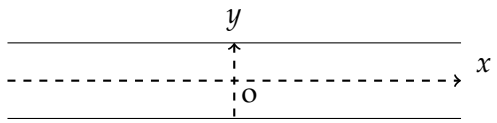
$$p_k = \varphi(p_1, ..., p_{k-1}).$$

Theorem 2.5. *If we choose F^i as specified above, and $G \equiv c$, then we have that $u_\epsilon \to \bar{u}$.*

The idea for the proof of Theorem 2.5 is already explained above. To explain the construction of v and c in Theorem 2.4, we start with a few examples.

Example 2.6. We consider the following simple junction problem in dimension 2 :

[5] which is defined up to an additive constant as for all cell problems.



The domain Σ

We are looking for v satisfying the following problem,

$$\bar{F}(x, D^2 v) = 0, \quad \frac{\partial v}{\partial n} = 0 \text{ on } \partial\Sigma,$$

such that

$$\lim_{x \to +\infty} \frac{v(x,y)}{x} = p_+,$$

and

$$\lim_{x \to -\infty} \frac{v(x,y)}{x} = -p_-.$$

We can find a solution that depends only on x. In this case, the equation takes the following form

$$\bar{F}\left(x, \begin{pmatrix} v_{xx} & 0 \\ 0 & 0 \end{pmatrix}\right) = 0.$$

Since $\bar{F}$ is strictly a non increasing function with respect to v_{xx}, and since $\bar{F}(x,0) = 0$, we have that

$$v_{xx} = 0,$$

so that

$$p_+ + p_- = 0,$$

which implies Kirchhoff's conditions in that particular case.

Example 2.7. In the case where the operator is simply a Laplacian Δ, Kirchhoff's conditions are obtained by integrating on a small volume Σ_R:

$$0 = \int_{\Sigma_R} -\Delta u = -\sum_{i=1}^{k} \int_{\Gamma_R^i} \partial_{x_i} u,$$

where $R > 0$.

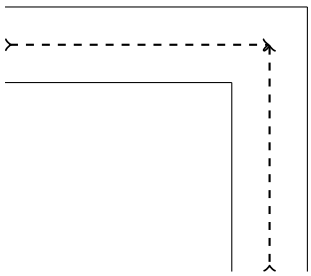
Example 2.8. We consider a problem in dimension 2 and we consider the following equation,

$$-\partial_{xx} u - \lambda\, \partial_{yy} u = 0,$$

where $\lambda \geq 0$, $\lambda \neq 1$. If we consider the junction of Example 2.6, then one has the same Kirchhoff relation

$$p_x + p_y = 0.$$

On the other hand, if we deform the tube on the following form,

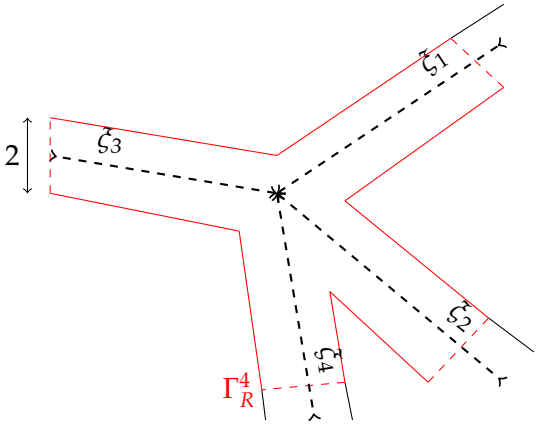


one gets the following relation

$$p_x + \lambda p_y = 0.$$

This example illustrates the importance of angles, and in general the geometry of the problem.

Proof of Theorem 2.4. Let us consider the following drawing to fix notations :



The domains Σ_1 *and* Σ_R

We consider the same domain Σ_1, we cut to the length $R > 0$ in order to get Σ_R, and we call Γ_R^i the section. The first step of the proof consists in introducing an approximate problem on Σ_R and prove that if $p_1, ..., p_{k-1}$ is fixed, then there exists a unique p_k^R which guaranties the existence of a unique function v_R (up to an additive constant) solution to the same equation, such that for all $1 \leq i \leq k-1$,

$$\frac{\partial v_R}{\partial n} = p_i \quad \text{on } \Gamma_i^R,$$

and

$$\frac{\partial v_R}{\partial n} = p_k^R \quad \text{on } \Gamma_k^R.$$

Note that this is a cell problem that can be solved by fixing a constant $c_R(p_1, ..., p_{k-1}, p_k^R)$. We shall see later why we must take this constant equal to zero. The second step is to take the limit when $R \to +\infty$ following the following steps : we show that if we cut at a level $0 < R_0 < R$, where R_0 is arbitrary, then v_R is locally bounded on Σ_{R_0} uniformly in R. Then, we explain why p_K^R is bounded and infer the behavior at infinity.

Without loss of generality, and up to a subsequence, one can always assume that

$$p_k^R \geq -C, \tag{2.8}$$

where $C > 0$, and we can replace v_R by $-v_R$ by changing all the signs.

From now on we simply use p_k instead of p_k^R to avoid distinguishing the different cases. Let us introduce two domains Σ_{R_0} and Σ_{R_1} analogous to Σ_R : a cut in R_0, and a cut in R_1, where $R_0 < R_1$. Since v_R is defined up to an additive constant, we can normalize v_R by requesting that

$$\inf_{\Sigma_{R_0}} v_R = 0. \tag{2.9}$$

Remark 2.9 (general remark). We have a second-order elliptic equation with Neumann conditions on the boundary of Σ_1. Using the maximum principle, we have for any $x > 0$,

$$\inf_{\Sigma_x} v_R = \min_{1 \leq i \leq k} \inf_{\Gamma_x^i} v_R.$$

Note that the above quantity defines a non-increasing function of x.

Remark 2.10. (a general bound) Let us consider one of these branches (say i), and let $(z, z') \in \mathbb{R}_+^2$ be such that $R > z' \geq z$, $|z|$ big enough to be in he flat part, and with analogous notations we consider the boundary Γ_z^i. One checks that

$$\sup_w v_R(z, w) + p_i(z' - z) \geq v_R(z', y) \geq \inf_w v_R(z, w) + p_i(z' - z). \tag{2.10}$$

To prove this bound, we use the maximum principle. Given the structure of the equation, the right and left sides in (2.10) are solutions of the equation and naturally satisfy Neumann conditions on the boundary because it doesn't depend on the transversal variable. Note also that these functions satisfies the conditions on Γ_R^i where $1 \leq i \leq k$. We infer by the maximum principle that v_R is between these two bounds. This is a simple argument which shows that if the function is bounded in a region, then we obtain the behavior at infinity automatically. Note that this property is strongly related to the fact that $\bar{F}(., 0) = 0$ (affine functions are solutions to the problem). We infer that once we solve the other questions, the behavior at infinity is obtained automatically.

Now we understand why we have to choose $c_R(p_1, ..., p_{k-1}, p_k^R) = 0$ among all the constants. In fact, it is the only constant which ensures that the right condition at the limit on the truncated domain is satisfied, and to avoid degeneration when we take the limit $R \to +\infty$.

We now show that v_R and p_R are locally bounded. Let us prove that v_R is bounded on Σ_{R_1} taking into account the normalization (2.9). Thanks to (2.10), one has

$$\inf_{\Gamma_{R_1}^i} v_R \geq \inf_{\Gamma_{R_1}^i} v_R + p_i(R_0 - R_1),$$

so that

$$\min_i \inf_{\Gamma^i_{R_1}} v_R \leq \min_i \inf_{\Gamma^i_{R_0}} v_R + C,$$

thanks to (2.8). It follows that

$$\inf_{\Sigma_{R_1}} v_R \leq \inf_{\Sigma_{R_0}} v_R + C,$$

thanks to Remark 2.9. The normalization (2.9) provides an homogeneous equation on Σ_{R_0} which is

$$\bar{F}(x, D^2 v_R) = 0, \quad v_R \geq 0.$$

Since

$$\inf_{\Sigma_{R_1}} v_R \leq C,$$

we infer from Harnack's inequality that

$$\sup_{\Sigma_{R_1}} v_R \leq C.$$

Note that the uniformly elliptic character is used here very strongly. Let us now prove that p^R_k are bounded. Note that for the branches $1, ..., k-1$ we have all the necessary : the locally bounded character and the behavior to infinity. In particular the following holds,

$$p_i x_i + C \geq v_R(x_i, y_i) \geq p_i x_i - C, \quad \forall y_i \quad \forall x_i \geq R_1.$$

In the branch k, it is enough to use (2.10) by introducing $R_1 < R_2 < R_0$. The above proof also applies for R_2 and we infer that v_R is bounded in Σ_{R_2}. Using (2.10), one has

$$\inf_{\Gamma^{R_2}_k} v_R \geq \inf_{\Gamma^{R_1}_k} v_R + p^R_k (R_2 - R_1),$$

which provides that

$$p^R_k \leq C.$$

This allows to prove that v_R is locally bounded everywhere, and moreover the asymptotic behavior follows using (2.10). The limit $R \to +\infty$ does not pose any problem: we take a subsequence such that the p^R_k converges and v_R converges uniformly on any bounded set to a v. Neumann conditions remain satisfied by the theory of viscosity solutions. One can also prove uniqueness of p^R_k and so we find the condition at the effective limit. □

We now specify a few points about the limit equations on the branches.

Remark 2.11. Consider the following equation on a branch of size ϵ,

$$F((x, y), D^2 u_\epsilon) + u_\epsilon = 0.$$

we know that $u_\epsilon \to u$ uniformly, where $u := \bar{u}$ is defined on one dimension, and we have on the boundary

$$\frac{\partial u_\epsilon}{\partial n} = 0.$$

Formally we expect to have an equation of the following form,

$$F\left((x, 0), \begin{bmatrix} u_{xx} & 0 \\ 0 & 0 \end{bmatrix}\right) + u = 0.$$

We prove this using the viscosity solutions theory. Let $\phi \in C^2$ and let us assume that the function $u(x) - \phi(x)$ has a strict local maximum in x_0. This implies that

$$u_\epsilon(x, y) - \phi(x)$$

have a local maximum in (x_ϵ, y_ϵ), where $x_\epsilon \to x_0$ and $|y_\epsilon| \leq \epsilon/2$, thanks to the uniform convergence. The difficulty is that the point (x_ϵ, y_ϵ) could be such that y_ϵ is on the boundary. In this case, we have not the claimed equation. To avoid this case, we will use a perturbation strategy, which will consist in introducing a perturbation in order to prevents the second coordinate to be on the boundary. More precisely, we consider the following quantity

$$u_{\epsilon,\delta}(x, y) - \phi(x) + \delta d(y),$$

where $d > 0$ inside the domain, is regular, and behaves like the distance to the boundary[6]: in particular we have that

$$d(\pm\epsilon/2) = 0.$$

Now, if the maximum point of $u_{\epsilon,\delta}$ is reached such that $|y_\epsilon| = \epsilon/2$ then

$$\partial_y u_{\epsilon,\delta}(x, y) = -\delta < 0,$$

which is a contradiction. We now have the following information on the equation,

$$F\left(x_\epsilon, y_\epsilon, \begin{bmatrix} \phi_{xx} & 0 \\ 0 & \mathcal{O}(\delta) \end{bmatrix}\right) + u(x_\epsilon, y_\epsilon) \leq 0.$$

Taking the limit when $\delta \to 0$ and $\epsilon \to 0$, we infer that

$$F\left(x, 0, \begin{bmatrix} \phi_{xx} & 0 \\ 0 & 0 \end{bmatrix}\right) + u(x_0) \leq 0,$$

which shows that it is a viscosity sub-solution. This explains how to get the limit equation.

Remark 2.12. If we consider the following equation on Σ_ϵ

$$F\left(x, \frac{x}{\epsilon}, Du_\epsilon, D^2 u_\epsilon\right) + u_\epsilon = 0,$$

the limit equations on the branches are the following,

$$F^i\left(x_i, (u_{x_i}, 0), \begin{bmatrix} u_{x_i x_i} & 0 \\ 0 & 0 \end{bmatrix}\right) + u = 0.$$

These equations are recovered using the same arguments presented above.

[6] for instance, one can consider the function $y \to \epsilon/2 - |y|$, regularized near zero.

2.2. **Hamilton Jacobi equations of the first order.** We consider the same kind of junctions and on each branch $x_i < 0$, we consider the following equation

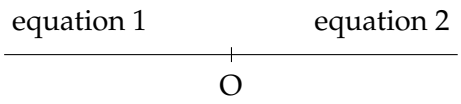
$$H_i(x_i, \partial_{x_i} u) + u = 0, \quad \text{and } u \in C(\Sigma).$$

In addition, we assume for instance that

$$\lim_{|z| \to +\infty} H^i(x_i, z) = +\infty,$$

uniformly in x_i.

Example 2.13. We consider the following particular case:



We start from the following equation,

$$H(x, Du_\epsilon) + u_\epsilon = 0 \quad \text{in } \Sigma_\epsilon,$$

where Σ_ϵ is the domain described in the previous subsections. Note that it is also possible to have

$$H\left(x, \frac{x}{\epsilon}, Du_\epsilon\right) + u_\epsilon = 0,$$

to manage the discontinuity, and we can also have a additional term on the form $-\eta\Delta$ in the previous equation. If $\epsilon = o(\eta)$, we can not see the regularization term. On the other hand, if $\eta = o(\epsilon)$, the additional term is useless. There is therefore a critical velocity when $\epsilon \sim \eta$.

The main novelty compared to the elliptic case is that in this case it is possible either to have Neumann conditions on the boundary or also state constraint conditions. This is a specificity of the first-order equations. An other difference with the elliptic equations is the following fact : in the elliptic case, whatever the value in zero is, we can always find a solution. This is not the case for first-order equations.

Another regularization problem consists in considering the following equation,

$$-\delta\Delta u_{\epsilon,\delta} + H(x, Du_{\epsilon,\delta}) + u_{\epsilon,\delta} = 0 \quad \text{in } \Sigma_\epsilon.$$

We get the following equation at the limit $\epsilon \to 0$,

$$\delta\, \partial^2_{x_i x_i} u_\delta + H_i(x_i, \partial_{x_i} u_\delta) + u_\delta = 0.$$

This is the case which has been treated up to now. Since we have a Laplacian, we recover a linear condition:

$$\sum_{i=1}^{k} \partial_{x_i} u_\delta(0) = 0.$$

and the terms of order one do not change the result. A natural idea of approximation of the idealized problem is to take the limit $\delta \to 0$ by the vanishing viscosity method.

Example 2.14. We now consider the following equation in dimension one :

$$(\partial_x u)^2 + u = 0.$$

In this case, $u \leq 0$ and therefore $u(0) \leq 0$. This example shows that we can not solve for any values of the function $c = u(0)$. Indeed, there are restrictions on the boundary conditions.

The previous example illustrates a typical aspect of first-order equations. Thus the natural question is: what is the set of c such that there exists a unique solution u continuous of viscosity such that $u(0) = c$. It is well known that if H is convex in z the set of possible values is given by $(-\infty, c_0]$ with $c_0 \in \mathbb{R}$. Note that the set of c which ensures the existence of solutions is also the set of c which ensures the existence of sub-solutions.

Moreover, this result is actually always true and does not require convexity. In fact : let c_0 be the value corresponding to the maximum solution. We want to show that the set of possible values c such that $c < c_0$ is an interval : which means that there is no holes. We take a possible value $c_1 < c_0$ and want to be sure that there are always solutions of values in (c_1, c_0). The idea is to use Perron's method : the supremum of sub-solutions is a super-solution. Let u_{c_1} be the solution corresponding to $u(0) = c_1$. Since, $c_1 < c_0$ this is not the state constraint solution, and therefore there exists a function ϕ such that

$$u_{c_1} - \phi$$

has a minimum in zero and

$$H\left(0, \frac{\partial \phi}{\partial x}(0)\right) + c_1 < 0.$$

Thanks to Perron's method there exists $\epsilon_0 > 0$, $\epsilon \in (0, \epsilon_0)$ and $\phi_\epsilon \in C^1$ such that

$$H(x, \partial_x \phi_\epsilon) + c_0 + \epsilon \leq 0.$$

Therefore, there exists a solution on $[c_1, c_1 + \epsilon_0)$. Note that by this method, one does not need the convexity.

Considering now the case of a junction with k branches, we have on each branch $i \in \{1, ..., k\}$ a value c_0^i, which corresponds to the equation on the branch k. Hence, to get a solution on all the branches, one has to choose

$$c \leq \min c_0^i.$$

Note that we can not do better: $c = \min c_0^i$ is the maximum solution because u is increasing with respect to c.

Proposition 2.15. *There exists a unique Lipschitz continuous solution of the k equations such that u is a super-solution at zero in the following sense: $\forall \phi \in C^1(\Sigma)$, $u - \phi$ has a local minimum in zero on Σ, and we have that*

$$\max\left(H_i(0, \partial_{x_i} \phi(0))\right) + u(0) \geq 0.$$

Furthermore

$$u(0) = \min_i c_0^i.$$

Proof. We use Soner's method for the state constraint problems on each of the half-lines ($x_i \leq 0$): if we have a super-solution and a sub-solution, we then know that the maximum of the difference will be reached at 0. We look at each branch of the junction, and the only difficulty is when the maximum point is at 0. In that case, we use the global condition assumed in the proposition to conclude. □

This result is recovered when we use the multi-dimensional approximation process (under technical assumptions) with state constraints on the boundary. At the limit $\epsilon \to 0$ we select this maximum solution. Moreover, if we consider the following approximation:

$$-\delta\, \partial^2_{x_i} u_\delta + H_i(x_i,\, \partial_{x_i} u_\delta) + u_\delta = 0.$$

We have formally $\lim_{\delta \to 0} u_\delta = u$ (converges up to a subsequence) provided that we impose a the right condition on boundary of type

$$c(\, \partial_{x_1} u, ..., \, \partial_{x_k} u) = 0,$$

where c is strictly increasing relative to each variable. When passing to the limit, we keep the equations and on each branch we have that

$$H_i(x_i,\, \partial_{x_i} u) + u = 0 \quad x_i < 0.$$

Uniqueness remains an open problem. We know that there is uniqueness if $H_i(0, z)$ has no flat part.

3. Seminar : New results on scalar conservation laws ?

3.1. **Introduction.** The conservation laws are important equations, because in physics it is quite natural to talk about preserved quantities. An interesting special case of conservation laws is the class of scalar conservation laws which are hyperbolic systems. One of the most famous examples is Burgers equation:

$$\partial_t u + \partial_x (u^2) = 0.$$

More generally, we are interested in equations on the following form

$$\partial_t u + \operatorname{div}(F(x,u)) = G(x,u), \tag{3.1}$$

where the unknown is a function defined from $(0,\infty) \times \mathbb{R}^d$ into $\mathbb{R}$, and the functions $F, G : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}^d$. Among conservation laws, the scalar case is simpler because we have at least formally a good behavior in the norm L^1. In fact, let u, v be two solutions to equation (3.1). If $G \equiv 0$ (for simplicity), we have that

$$\partial_t (u - v) + \operatorname{div}(F(x,u) - F(x,v)) = 0.$$

Thus, if we can write $F(x,u) - F(x,v) = A(x)(u-v)$, one can easily show that $t \to \|u(t) - v(t)\|_{L^1}$ is constant. On the other hand, a second peculiarity of this system is the comparison principle. In fact, at least formally if $u_0 \leq v_0$, we expect that $u \leq v$.

What is classical about these equations is the following:

- Shocks formation.
- Selection criteria for solutions: Oleinink.
- Selection criteria: Kruzkov's theory (L^1 framework).
- Idea of entropy.

Remark 3.1 (links with HJ equations). We consider the following equation in dimension one

$$\partial_t u + \operatorname{div}(H(x,u)) = 0.$$

If we define the function U as

$$U := \int_{-\infty}^{x} u,$$

then U satisfies the following HJ equation,

$$\partial_t U + H(x, \partial_x U) = 0.$$

If we have an L^1-framework for the first equation, then the framework of the second equation is C^0.

3.2. **Revisiting Kruzkov.** In the scalar case, it is relatively easy to find the entropies. In fact, consider the following equation in dimension one,

$$\partial_t u + \operatorname{div}(F(u)) = 0.$$

If we take $\eta' = F'\varphi'$, by change of variables, and at least formally, one has

$$J(\varphi) = \partial_t \varphi(u) + \partial_x \eta(u) = 0. \tag{3.2}$$

Thus, one expects that if $J = 0$ here, then as long as there is no shock we have that, $J(\varphi) = 0$, for every function φ. The entropy criterion consists in asking that $J(\varphi) \leq 0$, for all convex functions φ.

If we consider J on the set of convex function, then one can look only at the extreme points since J is linear, and all the other convex functions are convex combinations of these extreme points. The extreme points are $|u-k|$, which corresponds to the fact that φ'' is a Dirac mass. We also look at $(u-k)^+$, and $(k-u)^+$. These functions generate the set of increasing convex functions and decreasing convex function respectively.

By analogy with Kruzkov's theory, we have been able to understand well HJ equations in the sense of viscosity, and we would like to do the opposite now. Let us start by recalling Kruzkov's theorem. Let $u \in L^1$ be an entropic solution to the following system

$$\partial_t u + \operatorname{div}(F(u)) = 0 \quad u_{t=0} = u_0.$$

Theorem 3.2 (Kruzkov)**.** *For any $u_0 \in L^1$, there exists a unique entropic solution.*

Sketch of the proof. The key point in Kruzkov's proof is the idea of doubling the variables. In fact, if u, v are two solutions, we consider $u(x,t) - v(y,s)$. One can think of a stationary problems of the following form, to simplify the presentation[7] :

$$u + \operatorname{div}_x(F(u)) = u_0$$

$$v + \operatorname{div}_y(F(v)) = v_0$$

When we take for example $\varphi = (u-k)^+$, the η that works is

$$\eta(u) = (F(u) - F(k))\operatorname{sign}_+(u-k).$$

[7] by integrating in time for instance.

Using the entropy relation (3.2) above (with $k = v(y)$ and $k' = u(x)$), we have that

$$\begin{aligned} |u(x) - v(y)| + \partial_x\{[F(u(x)) - F(u(y))]\,\mathrm{sign}(u-v)\} \\ + \partial_y\{[F(u(x)) - F(u(y))]\,\mathrm{sign}(u-v)\} \\ \leq (u_0(x) - v_0(y))\,\mathrm{sign}(u_0 - v_0) \leq |u_0 - v_0|. \end{aligned} \tag{3.3}$$

Now, what we want to recover is the information in x and so we bring y closer to x, using a suitable kernel. More precisely, we multiply inequality (3.3) by $x \to \rho_\varepsilon(x-y)$ where

$$\rho_\varepsilon := \frac{1}{\varepsilon^d}\rho(\frac{\cdot}{\varepsilon}), \quad \rho \geq 0, \quad \int \rho = 1, \quad \text{and} \quad Supp(\rho) \text{ is compact.}$$

Integrating by part, one obtains that

$$\int\int |v(x) - v(y)|\rho_\varepsilon(x-y) \leq \int\int |u_0 - v_0|\rho_\varepsilon(x-y).$$

Thus, taking the limit $\varepsilon \to 0$, we get the L^1 norm, namely

$$\|u - v\|_{L^1} \leq \|u_0 - v_0\|_{L^1}.$$

This shows not only uniqueness, but also stability and the contraction effect. □

It is also possible to define entropic sub-solutions and entropic super-solutions. In fact, we say that u is an entropic sub-solution, if

$$\partial_t(u-k)_+ + \mathrm{div}\{(F(u) - F(k))sign_+(u-k)\} \leq 0.$$

This is equivalent to say that we have the entropy inequality for all increasing convex functions. This implies that u is a sub-solution in the sense of distributions for the following equation

$$\partial_t u + \mathrm{div}(F(u)) \leq 0.$$

In the same way, we define super-solutions, through the entropy inequality for all decreasing convex functions.

Using the same ideas of Kruzkov's theorem proof, one can look what happens for u if we use $(u-k)^+$ and $(k-u)^+$ for v. One checks that

$$\begin{aligned} (u(x) - v(y))^+ + \mathrm{div}_x\{(F(u) - F(v))\,\mathrm{sign}_+(u-v)\} \\ + \mathrm{div}_y\{(F(u) - F(v))\,\mathrm{sign}_+(u-v)\} \leq (u_0 - v_0)^+. \end{aligned}$$

This allows to prove that if $u_0 \leq v_0$ then $u \leq v$ for solutions, and also for sub-solutions and super-solutions. Hence, this gives a comparison principle, and shows that sub-solutions are below the natural solutions.

Remark 3.3. i) Perron's principle is satisfied : the solution is the supremum of all sub solutions.

ii) Now, we are interested in a very useful notion in the framework of HJ equations : inf-convolution and sup-convolution of continuous functions. We use notations of Remark 3.1. The most often used sub-convolution for HJ equations is the following

$$\sup_y \left\{ U(y) - \frac{1}{2\varepsilon}|x-y|^2 \right\} = \sup_y \left\{ U(x-y) - \frac{1}{2\varepsilon}|y|^2 \right\}.$$

Now, the question that naturally arises is : it is possible to use this tool for scalar equations using Remark 3.1? Since the invariance by translation is not adapted to the scalar equations we change the definition and consider the following:

$$\sup_{|x-y|\leq\varepsilon} \{U(y)\}.$$

Now to adapt, we start from an entropic sub-solution for the conservation equation and consider

$$\Psi(x) := \sup_{|x-y|\leq\varepsilon} \{u(y)\} = \sup_{|y|\leq\varepsilon} \{u(x-y)\} \tag{3.4}$$

assuming that we have enough regularity in order to give sense to the previous expressions. Now, the natural question that arises is the following: is the sup of two entropic sub-solutions a sub-solution? The answer is yes. In fact, Kruzkov's Theorem tells us more than the simple fact of uniqueness, it also says that the sup of two entropic sub-solutions is an entropic sub-solution. By adding constants in the definition of an entropic sub-solution, we have that

$$\partial_t \max(u,k) + \operatorname{div}_x(F(\max(u,k))) \leq 0.$$

In the same way, the solution can be written in the following form

$$\partial_t \min(u,k) + \operatorname{div}_x(F(\min(u,k))) \geq 0.$$

Note the similarity with the notions of sub-solutions and super-solutions for HJ equations, except that here the functions test are $\max(u,k)$ and $\min(u,k)$.

Theorem 3.4. *If u and v are entropic sub-solutions, then $\max(u,v)$ is an entropic sub-solution.*

Sketch of the proof. We use the same idea of Kruzkov's theorem, by doubling the variables and considering $\max(u(x),v(y))$. We use the previous inequalities, and the same argument for

$$\int \rho_\varepsilon(x-y)\left\{\operatorname{div}_x(F(\max(u(x),v(y)))) + \operatorname{div}_y(F(\max(u(x),v(y))))\right\}.$$

The first term vanishes completely. □

In view of these remarks, the regularization (3.4) is well defined and in particular it preserves the aspect of sub-solution thanks to Theorem 3.4. If we know how to define this regularization process, then every positive solution is in the function space BV.

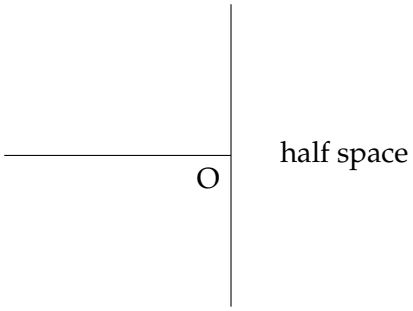
Indeed, regularization (3.4) turns to solve the following equation with respect to ϵ,

$$\partial_\varepsilon u + |\nabla u| = 0.$$

Hence, at least formally, if we have L^1 regularity, then we get BV regularity by integrating. We only need to pay attention to the fact that if we have L^1 regularity, then the supremum in the definition (3.4) is only an essential supremum. In this case we cannot say that Ψ is a sub-solution because the essential sup is not necessarily a limit of countable sup. Thus, one can always make the regularization, but it does not always converge to a function: it converges to an envelope that has a particular property. However, for a function that is reasonable in terms of L^1, the definition gives a regularization BV which preserves the notion of sub-solution.

3.3. **Boundary conditions for conservation laws.** In this subsection, we explore the equivalence for conservation laws of the notion of state constraints (for HJ equations). For conservation laws, this means that what happens at the boundary will depend mainly on the signs. Looking at these extreme cases, we have Dirichlet boundary conditions.

3.3.1. *Saturated solutions.* For saturated solutions, all signs count. Think of a half line that ends in 0 with a half-space (in general, a hyperplane):



A saturated solution in the half-space $x_1 \geq 0$ is an entropy solution on the half-space which is an entropic super-solution on the boundary for some k.

This means that the inequality is written on the closure of the set: multiply by test functions and integrate by parts on the closure of the domain, and we recover $F(k)$ as a boundary term. We prove that there exists a unique solution which satisfies the above conditions and that is greater than all the entropic solutions on the second domain provided that $F_1(z) \to +\infty$ if $|z| \to +\infty$. The terminology of saturated solution comes from this last property.

3.3.2. *Dirichlet boundary conditions.* Consider a domain Ω and assume that $u_{\partial\Omega} = u_0 \in L^1$. We have the following new problem,

$$\partial_t u + \mathrm{div}_x(F(x,u)) = 0 \quad u_{\partial\Omega} = 0.$$

Here the x dependence in the transport term is $W^{1,1}$. In this problem, we have an over-determination and therefore we must relax the problem. We define an entropic sub-solution in the following way:

$$\forall k > 0, \quad \forall \varphi \in C^1(\overline{\Omega}), \quad -\int_\Omega F(\max(u,k)).\nabla\varphi + \int_{\partial\Omega} F(k).n \leq 0. \tag{3.5}$$

In that case, we can prove existence and uniqueness of a solution. The result is proved using a regularization by viscosity by considering the following problem,

$$-\varepsilon\Delta u + \partial_t u + \mathrm{div}_x(F(x,u)) = 0, \quad u^\varepsilon_{\partial\Omega} = 0.$$

For this problem, there is no more over-determination and the solution is regular. We consider the following quantity:

$$\max(u^\varepsilon, k)_{\partial\Omega} = k$$

which is constant and equal to k in the neighborhood of the boundary. Thus, when integrating by parts the Laplacian is zero and the other terms are exactly the terms in definition 3.5.

⋆ ⋆ ⋆ ⋆

APPENDIX A. COURSE DATES AND SECTIONS

- Lecture of October 21th 2016 : General introduction; Example 1.1; Example 1.2; Lemma 1.3;
- Lecture of November 4th 2016 : Remark 1.4; Section 1.1.2-The naive approach; Section 1.1.2-Partial information approach;
- Lecture of November 18th 2016 : Section 1.1.2-Reduction to finite dimension; Example 1.6; Section 1.1.3;
- Seminar of November 18th 2016 : Section 3;
- Lecture of November 25th 2016 : Example 1.10; Section 1.2.1; Section 1.2.2;
- Lecture of December 2nd 2016 : Section 1.2.3; Section 1.2.4;
- Lecture of December 9th 2016 : Section 1.2.5; Section 1.2.6;
- Lecture of December 16th 2016 : Section 1.2.7; Introduction of Section 2;
- Lecture of January 6th 2017 : Proposition 2.1; Example 2.3; Theorem 2.4; Theorem 2.5; Example 2.6; Example 2.7; Example 2.8;
- Lecture of January 13th 2017 : Proof of Theorem 2.4; Remark 2.9; Remark 2.10; Remark 2.11; Remark 2.12; Section 2.2;

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