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René Aïd, Dylan Possamai & Nizar Touzi

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René Aïd* Dylan Possamaï† Nizar Touzi‡

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Abstract

Despite the success of demand response programs in retail electricity markets in reducing average consumption, the literature shows failure to reduce the variance of consumers' responses. This paper aims at designing demand response contracts which allow to act on both the average consumption and its variance.

The interaction between the producer and the consumer is modeled as a Principal-Agent problem, thus accounting for the moral hazard underlying demand response programs. The producer, facing the limited flexibility of production, pays an appropriate incentive compensation in order to encourages the consumer to reduce his average consumption and to enhance his responsiveness. We provide closed–form solution for the optimal contract in the case of linear energy valuation. Without responsiveness incentive, this solution decomposes into a fixed premium for enrolment and a proportional price for the energy consumed, in agreement with previously observed demand response contracts. The responsiveness incentive induces a new component in the contract with payment rate on the consumption quadratic variation. Furthermore, in both cases, the components of the premium exhibit a dependence on the duration of the demand response event. In particular, the fixed component is negative for sufficiently long events. Finally, under the optimal contract with optimal consumer behaviour, the resulting consumption volatility may decrease as required, but it may also increase depending on the risk aversion parameters of both actors. This agrees with standard risk sharing effects.

The calibration of our model to publicly available data of a large scale demand response experiment predicts a significant increase of responsiveness under our optimal contract, a significant increase of the producer satisfaction, and a significant decrease of the consumption volatility. The stability of our explicit optimal contract is justified by appropriate sensitivity analysis.

*Université Paris–Dauphine, PSL Research University, LeDA. The author received the support of the Finance for Energy Markets Research Centre and of the ANR CAESARS ANR-15-CE05-0024-02. rene.aid@dauphine.fr. This work benefited from comments at the SIAM Conference on Financial Mathematics & Engineering, Austin (2016), at the 10$^{\text{th}}$ Bachelier World Congress, Dublin (2018). A vulgarization of the model presented in this paper was published in *SIAM News*, June 2017.

†Columbia University, IEOR, 500W 120th St., 10027 New York, NY, dp2917@columbia.edu. Research supported by the ANR project PACMAN ANR-16-CE05-0027.

‡École Polytechnique, CMAP, nizar.touzi@polytechnique.edu

1 Introduction

Part of the equation to achieve the COP21 objective, of limiting the climate change effects to a 2 degrees Celsius increase, relies on the design of carbon-free electric systems. According to the International Energy Agency 2016 report on carbon emission from fuel combustion more than a third of carbon emission for energy system in the world comes from power generation (source IEA [24]). The massive development of renewable energy sources worldwide, mainly solar and wind power is helping reaching this objective. But, at the same time, they are reshaping the way power system have to be managed. Renewable energy sources are intermittent and not dispatchable. The increase in the uncertainty of generation has made the question of flexibility at the heart of the design of power systems with a large penetration of renewable energy sources.

In this paper, we focus on the use of demand response contracts to achieve flexible power systems. A demand response mechanism is a contract under which the consumer benefits from cheaper electricity than the standard tariff, and accepts in turn to suffer from much higher prices at certain peak load periods chosen by the producer. These *soft* mechanisms appear to cumulate the virtues of consumption reduction, while providing substitutes to *hardware* technologies as chemical batteries or flexible gas–fired plants. Their existence and use is crucial to achieve reliability and efficiency of power systems, see the related work by Joskow and Tirole (2007) [26]. Because of the stakes involved, a revival of experiments has flourished worldwide to precisely assess the effects of demand response programs on end–users' consumption, see Abrahamse et al. (2005) [1], Herter (2007) [18], Faruqui and Sergici (2010) [16], Newsham and Bowker (2010) [34], Wolak (2011) [43], Jessoe and Rapson (2014) [25]. For this reason, many OECD countries are making significative investment in the development of smart-meters as a key technology for new generation demand-side management programs. This is best witnessed by the 45 millions smart meters deployed in Italy, Swede and Finland, and the overall EU-27 objective of 200 millions smart meters, that is 225 € per consumer[1].

Nevertheless, demand response mechanisms have some challenges to overcome before they can pretend to provide a level of flexibility comparable to gas–fired plants or chemical batteries. First, the incentive scheme needs to be devised so as to avoid *baseline manipulation*, see Chao and De Pillis (2013) [10] in the context of the Baltimore stadium management by Enerwise company case[2]. This raises the general question of moral hazard in the present context, which we address in the present paper by modeling the interaction between the consumer and the producer through a Principal Agent problem.

Our second main contribution is to address the well documented fact that demand response programs exhibit a substantial variance in the response of consumers to price signal. This leads to uncertainty on the total response of the solicited population, see Carmichael et al. (2014) [8, Section 4.3], and Section 2 below for the motivating facts of our work. This large variance is called the *responsiveness* effect, and stands as a key deficiency of demand response programs. Jessoe and Rapson (2014) [25] investigate the reasons for this poor responsiveness, and points out the role of communication technologies and information. The main objective of this paper is to device incentive mechanisms in order to enhance the responsiveness of consumers and to provide estimates of the potential gains they may induce.

We formulate the mechanism design of demand response programs as a problem of continuous–time optimal contracting between a producer and a consumer. A risk–averse CARA producer has to satisfy the random electricity consumption of a risk–averse CARA consumer during a given period of time. We focus on the consumption deviation relative to a predictable pattern of the consumer's demand. The producer has a generation cost function for the energy and is also subject to a direct cost of the consumer's responsiveness

[1] Report form the Commission *Benchmarking smart metering deployment in the EU-27 with a focus on electricity*, SWD(2014) 188 final, p. 4.

[2] Enerwise was fined a $780.000 penalty by the Federal Energy Regulation Commission 143 FERC 61218 as of June 7th, 2013 for manipulation of a demand response program.

defined as the consumption volatility. The consumer has a subjective value of energy and can reduce both the average level of his consumption and its volatility by taking costly actions which may depend on the nature of the corresponding usage of electricity (lights, oven, air conditioning, tv, computer...). The producer only observes the total consumption of the consumer and has no access to the consumer's actions or efforts. She aims at finding the optimal contract that minimises the expected disutility from the energy generation cost, the responsiveness cost and the incentive payment, while anticipating the optimal response of the consumer's maximisation of his expected utility from the payment, the benefit value of his deviation and the cost of efforts. Finally, the producer's problem is subject to the consumer's participation level defined as the reservation utility without contract.

The execution of the contract described above can be interpreted as a price event of a dynamic tariff. With this interpretation, the reduction of the volatility of consumption on a single exercise is similar to an increase of consumer's responsiveness. If there is a reduction in the volatility of consumption, the producer will observe a lower variance of consumption reductions across price events. Thus, from now on, we will use indifferently the terms of *volatility reduction* or of *responsiveness increase*. It should be noted that the interpretation we use in this paper of the reduction of volatility as an increase in the responsiveness of the consumer applies to other Principal-Agent situations.

We solve the first–best and second–best optimal contracting problems in closed form in the context of linear consumer's energy value and producer's energy generation cost. The optimal contract consists in a deterministic payment that depends on the the duration of the demand response, a linear payment on each infinitesimal deviation, and a linear payment on the realised squared volatility. We find that the risk–sharing process depends on the difference between the value of energy for the consumer and the energy generation cost of the producer. This difference corresponds to peak period when energy is more costly to produce than it has value for the consumer and off-peak periods when it is the opposite.

In agreement with the economic intuition, we find that the producer induces efforts to reduce the average consumption only on peak periods when energy is more costly to generate than it has value for the consumer. This result is consistent with demand response programs which target peak periods, see Faruqui and Sergici (2010) [16]. Further, we find that the responsiveness incentive does not depend on the period of the day, but on the marginal responsiveness cost of the producer. The latter decomposes into the direct linear responsiveness cost and a risk–premium accounting for the risk–sharing between the producer and the consumer. The responsiveness incentive is active even without direct responsiveness cost in the producer's criterion.

As a consequence, under the optimal contract with optimal consumer behaviour, the resulting responsiveness may improve as required, but it may also deteriorate depending on the risk aversion parameters of both actors. Because he is risk–averse, the consumer has a natural incentive to reduce the consumption volatility, even without contract. Thus, before contracting one could observe a higher level of responsiveness than the no-effort situation. After contracting, if the producer's responsiveness cost is small enough, or if she is much less risk–averse than the consumer, she would bear enough risk thus decreasing the burden of the consumer to improve responsiveness. This explains the possible responsiveness decrease, and illustrates how the risk sharing effect induces that the electric system can bear more risk.

Another result of our model is that, during off–peak periods, the producer's first–best value can be implemented by second–best optimal contracting, with identical optimal contracts. This result is not valid anymore during peak periods: the consumer does enjoy a positive information rent for sufficiently small consumer's responsiveness cost of effort. This is due to the fact that the incentives on level reduction and responsiveness improvement are combined in this context, while they act separately in the first best contracting situation. This mixed effect is responsible for the loss of value from the producer's side.

We next turn to the empirical examination of our model. We use the publicly available data of the large scale demand response experiment of Low Carbon London Pricing Trial, see Section 7 for a description of the data. The experiment allows to compare a controlled group of consumers enjoying a standard flat tariff with a group of consumers enrolled with a dynamic Time–of–Use tariff (dToU). We calibrate the parameters of our model by interpreting this experiment as the implementation of our optimal contracting model under no responsiveness incentives, i.e. the consumer is only incentivised to reduce the average level of his consumption deviation.

Using this hypothesis, we find that the implementation of a responsiveness incentive would result in a reduction of volatility of 17% and a multiplication by three of the producer certainty equivalent compared to an optimal contract based only on incentives on average consumption reduction. The benefit of the producer would increase from 0.33 pence per price event and consumer to 1 pence. Whether or not the responsiveness incentive is active, we find that the optimal payments are positive, non–monotonic and increase for large values of the demand response events duration. But, while the deterministic and random part payments without responsiveness control are always positive, we find that the responsiveness control induces a negative deterministic payment combined with a large positive certainty equivalent random payment. The responsiveness incentive consists in, first taking cash out of the consumer, and then giving him back a potential large payment for appropriate realisation of the output.

We also find a significant cost of action for the consumers to reduce their consumption of approximately 18 p/KWh, a value that ranges between the normal day price of 11.42 p/KWh and the high price during events of 67.2 p/KWh. But, as the LCL Pricing Trial did not involve consumers enroled with a real–time pricing tariff (RTP), we can not compare this cost to the cost induced by RTP as Wolak (2011) [43] does. Nevertheless, our result provides new insights in the debate regarding the incentive policies proposed by the Federal Energy Regulatory Commission to foster the development of demand response programs (FERC Order 745). The design of efficient incentive policies to foster consumer's demand response as proposed by the FERC can not rely on the marginal cost of electricity of the producer, or only on the difference between this marginal cost and the value paid by the consumer as suggested by Hogan (2009,2010) [20, 21], Chao (2011) [9] and Brown and Sappington (2016) [6], but should also take into account the disutility incurred by the consumer for his efforts to induce the reduction as well as the risk–aversion of both the consumer and the producer even when there is no responsiveness control implemented.

Finally, we examine the robustness of our results to our linearity assumption which is crucial for the derivation of the explicit solution. We implement a numerical approximation of the solution of the optimal contracting problem for a concave consumer's energy value function, and we perform the comparison with the corresponding linear contract approximation. We find that the more the energy value is concave, the more the producer overpays the consumer. But, even for large concavity value, the linear contract succeeds in providing 80% of the benefit issued from the optimal second–best contract. Further, the responsiveness control mechanism still provides a significant benefit to the producer. The pure effect of concavity leads to a decrease of the gain to 70% while the joint effects of concavity and multiple usages leads to a decrease of the gain to 30%.

Our model stands at the intersection of optimal contract theory in continuous-time, the economics of demand response for power systems and theirs empirical implications. From a methodological point of view, our work falls in the line of the works of Hölmstrom and Milgrom (1979) [22], Grossman and Hart (1983) [17], Hölmstrom and Milgrom (1987) [23] and Sannikov (2008) [36]. Our results rely on the recent advances of Cvitanić et al. (2018) [13] which allows volatility control (i.e. responsiveness effort) in the continuous time Principal Agent problem. For an economic introduction to incentives theory, we refer to Laffont and Martimort (2002) [30]. The economic literature on optimal contract theory and risk-sharing is vast. We also refer to Cadellinas et al. (2007) [7], who present a general setting for the analysis of the first–best optimal risk–sharing in the context of compensation plan for executives, and to Müller (1998) [33] who provides the first–best optimal sharing rule in the case of CARA exponential utility function and shows that it is also a linear function of the aggregated output as in the second–best case. The closest work on volatility control in the framework of Principal–Agent is in Cvitanić et al. (2017) [14], however our paper is the first to produce a closed form solution in this context.

Regarding demand response programs, its study is a long dated subject in the economic literature, see Tan and Varaiya (1993) [38] for a seminal model of interruptible contracts for a pool of consumers. The framework of optimal contract theory has been used to formulate the enrolment of customers in demand response programs as an adverse selection program. This idea can be traced back to the work of Fahrioglu and Alvarado (2000) [15] on incentive compatible demand response program and has been recently reused both in the work of Crampes and Léautier (2015) [12] to design suitable base–line consumption reference, and by Alasseur et al. (2017) [2] for peak–load pricing. More recently, big data methods have been used to provide precise response of consumers with a given probability, see Kwac and Rajagopal (2014) [29]. The only known work to the authors modeling the cost of electricity demand volatility is Tsitsiklis and Xu (2015) [40] who designed a model in discrete–time where they take into account not only the generation cost of energy but also the cost of variation of generation between two–time steps. In their model, consumers are incited to reduce their consumption with a price signal which is the traditional marginal fuel cost of generation plus the marginal of cost of variation of generation. The complexity of the model in terms of represented constraints only allows for numerical simulations.

Finally, our work contributes to the empirical literature on moral hazard models, see Chiappori and Salanié (2000) [11] for insurance industry, Lewis and Bajari (20014) [31] for public procurement and Bandiera et al. (2005) [3] for workers productivity. We make the testable prediction that the implementation of responsiveness incentives decreases consumption volatility. This claim can be tested by the next generation of demand response pricing trials by performing controlled experiments to compare consumers with and without responsiveness incentives as described in our work.

The paper is organised as follows. Section 2 provides evidences that current demand response mechanism fails to increase consumer's responsiveness, i.e. to decrease the volatility of his responses. Section 3 describes the model. Section 4 provides the optimal contract and effort of the first–best problem in the general case. Section 5 provides the optimal contract and effort characterization in the second best case. Section 6 gives the closed-formed expression of the optimal contract and efforts in the case of a linear energy value and generation cost and compares the first– and the second–best problem. Section 7 provides empirical results. Section 8 concludes.

2 Motivating facts

In this section, we provide evidence that consumers exhibit a large variance in their response to standard dynamic pricing schemes. Our argument relies on the existing literature on one hand, and on an empirical examination of the Low Carbon London Pricing Trial. Our main concern is that these programs fail to enhance the responsiveness of consumers.

The first objective of demand response programs is to incite consumption reduction during peak–load periods. Significant experiments have been conducted worldwide to assess the efficiency of such programs. Faruqui & Sergici (2010) documents more than 30 trials which conclude to a positive effect of demand response incentives. However, an important concern is about the responsiveness of the consumers. The efficiency of demand response programs to achieve average consumption reduction comes with an irregular response of the consumers during the pricing trials. Figure 2 shows the average consumption reduction of more than 25 pricing trial. The reduction of consumption in peak-load exhibits a large variance with an efficiency ranging from 10% to 50%. This variance is still large in the population enrolled in Critical Peak Pricing with digital communication devices (CPP w/Tech).

We next use the Low Carbon London Pricing Trial publicly available data to illustrate that dynamic Time-of-Use tariff (dToU) fails to control the enrolled consumers responsiveness, although the overall effect of the incentive mechanism is positive. In the Low Carbon London Pricing Trial, a group of consumers enjoying a standard flat tariff was used as a control group while a group of consumers was enrolled in a

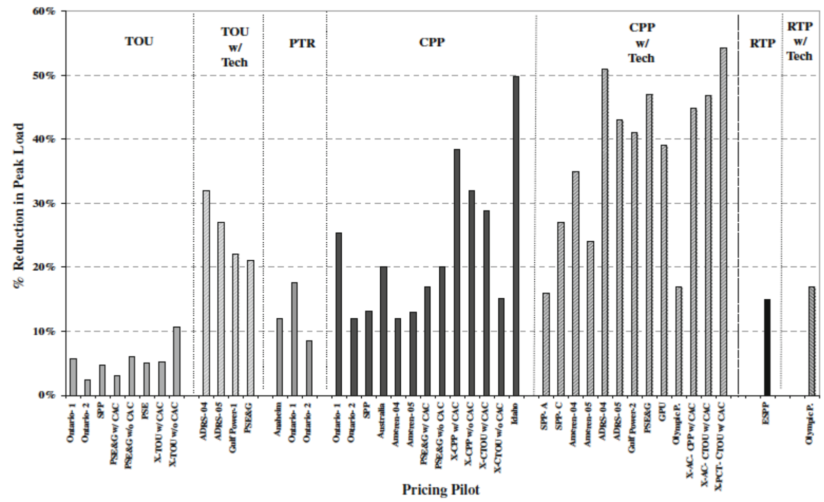

Figure 1: Household response to dynamic pricing of electricity, source: Faruqui & Sergici (2010) [16].

dynamic Time–Of–Use tariff. Tariffs were sent to the latter households on a day–ahead basis using a Home Display or a text message to the mobile phone of the customer. A price event is a day when a high price is sent to the dToU group. The precise description of the dToU trial performed in 2013 is given in [39, chap. 3] and will be discussed further in Section 7. The Low Carbon London Demand–Side Response Trial was designed to be as close as possible to a random trial experiment, while accounting for the operational constraints due to the enrollment of a large set of customers within the portfolio of given UK utility (EDF Energy). In the following, the size of the samples is 880 consumers for the control group and 250 for the dToU group and the number of price events is 69.

We decompose the consumption of each household i as $C_t^i := S_t^i + X_t^i$, where S_t^i is a deterministic seasonal component and X_t^i is the deviation of the household consumption to the deterministic pattern S_t^i. In order to compare the two populations, we formulate the following linear model

$$X_t^i = m + v\,\varepsilon_t^i, \;\; \text{if} \;\; i \;\; \text{enrolled}, \;\; \text{and} \;\; X_t^i = m' + v'\varepsilon_t^i, \text{if} \;\; i \;\; \text{not enrolled},$$

where m, m' and $v, v' \geq 0$ are unknown parameters, and ε_t^i are independent centered with unit variance.

These parameters are naturally estimated by the means $\hat{m}, \hat{m}'$ and the standard deviations $\hat{v}, \hat{v}'$ within each population. We next perform the test of the hypothesis that $m = m'$ and $v = v'$. The results are reported in Table 1. The last column provides the p-value of the Student test for the equality of the mean and of the Fisher-Snedecor test for the equality of the standard deviation of the two samples. The equality of the means is statistically rejected with a high level of confidence, indicating that there is an average reduction effect of approximately 10 Watt. However, again with a high level of confidence, the equality of standard deviations can not be rejected, indicating that the program has no effect on the responsiveness.

The purpose of this paper is to provide optimal incentives to induce both reduction of the average consumption of the consumer and to enhance his responsiveness, i.e. to reduce the volatility of his consumption during price events. We shall provide predictions of the potential benefits from responsiveness incentive. The calibration performed in Section 7 of the Principal–Agent model described in Section 3 with the data of the LCL pricing trial shows that during peak–load periods the implementation of a responsiveness incentive

– can increase by more than 200% the gain of the producers; in the situation defined by the calibration of the model, responsiveness incentive induces an increase of the producer's certainty equivalent from 0.33 to 1.03 pence;

– reduce the volatility of consumption by 20%.

	not enrolled i.e. standard	enrolled i.e. dToU	Test Hypothesis p-value
Mean	13.9	3.5	$m = m'$: 1.3 10^{-15}
Standard deviation	87.2	66.2	$v = v'$: 0.128

Table 1: Mean and standard deviation of consumption deviation for the Standard tariff group and for the dToU tariff group during price events (unit Watt).

3 The model

The Agent consumption deviation process is denoted by $X = \{X_t, t \in [0,T]\}$, and is the canonical process of the space Ω of scalar continuous trajectories $\omega : [0,T] \longrightarrow \mathbb{R}$, i.e. $X_t(\omega) = \omega(t)$ for all $(t,\omega) \in [0,T] \times \Omega$. We denote by $\mathbb{F} = \{\mathcal{F}_t, t \in [0,T]\}$ the corresponding filtration.

3.1 The consumer

A control process for the consumer (the Agent) is a pair $\nu := (\alpha, \beta)$ of $\mathbb{F}-$adapted processes, which are respectively $A-$ and $B-$valued. More specifically, α represents the effort of the consumer to reduce the nominal level of consumption and β is the responsiveness effort consisting in reducing the consumption deviation variability for each usage of electricity. We emphasize that α and β are respectively $N-$ and $d-$dimensional vectors, thus capturing the differentiation between different usages, e.g. refrigerator, heating or air conditioning, lightning, television, washing machine, computers... This set of control processes is denoted by $\mathcal{U}$.

For a given initial condition $x_0 \in \mathbb{R}$, representing the current deviation from the deterministic part of consumption, and some control process $\nu := (\alpha, \beta)$, the controlled equation is defined by the following stochastic differential equation[3] driven by an $d-$dimensional Brownian motion W

$$X_t = x_0 - \int_0^t \alpha_s \cdot \mathbf{1} ds + \int_0^t \sigma(\beta_s) \cdot dW_s,\ t \in [0,T],\ \text{with } \sigma(b) := (\sigma_1 \sqrt{b_1}, \dots, \sigma_d \sqrt{b_d})^\top, \tag{3.1}$$

for some given parameters $\sigma_1, \dots, \sigma_d > 0$. All of our utility criteria depend only on the distribution $\mathbb{P}^\nu$ of the state process X corresponding to the effort process ν. Let $\mathcal{P}$ be the collection of all such measures $\mathbb{P}^\nu$.

The state variable X represents the consumer's deviation from the deterministic profile of his consumption. An effort ν induces a separable cost $c(\nu) := c_1(\alpha) + \frac{1}{2} c_2(\beta)$ and a utility $f(X)$ from the deviation X. Throughout this paper, we shall take

$$c_1(a) := \frac{1}{2} \sum_{i=1}^N \frac{a_i^2}{\mu_i},\ \text{and } c_2(b) := \sum_{j=1}^d \frac{\sigma_j^2}{\lambda_j \eta_j} \big(b_j^{-\eta_j} - 1 \big),\ a \in A,\ b \in B,$$

[3] For technical reasons, we need to consider weak solutions of the stochastic differential equations. However, for expositional purposes, we deliberately ignore this technical aspect in this section so as to focus on the main message of the present paper.

for some $(\mu, \lambda, \eta) \in (0, +\infty)^N \times (0, +\infty)^d \times (1, +\infty)^d$. Notice that c is convex, increasing in a and decreasing in b as the responsiveness effort consists in reducing the volatility, thus reproducing the requested effects of increasing marginal cost of effort. Moreover, $c_1(0) = c_2(1) = 0$ captures the fact that there is no cost for making no effort. The cost function is quadratic in a, and illustrates that small deviations (as e.g. switching off the light when leaving some place) are painless, while large deviations (as e.g. reducing the consumption from heating or air conditioning) are more costly. The function f is increasing, and centered at the origin. A specific choice of f is not important for the general presentation, but closed form solutions will be obtained for linear f.

The Agent controls the electricity consumption deviation by choosing the effort process ν in the state equation (3.1), subject to the corresponding cost of effort rate $c(\nu)$, and the deviation utility rate $f(X)$. For technical reasons, we need to consider bounded efforts, we then set

$$A := \big[0, \mu_1 A_{\max}\big] \times \cdots \times \big[0, \mu_N A_{\max}\big] \;\text{ and }\; B := [\varepsilon, 1],$$

for some constants $A_{\max} > 0$ and $\varepsilon > 0$.

The execution of the contract starts at $t = 0$. The consumer receives a payment ξ from the producer at time T, in compensation for his consumption deviation. The producer has no access to the consumer's actions, and does not observe the consumer's different usages of electricity and deviations. She only observes the overall deviation X. Consequently, the compensation ξ can only be contingent on X, that is ξ is $\mathcal{F}_T-$measurable. We denote by $\mathcal{C}$ the set of $\mathcal{F}_T-$measurable random variables. The objective function of the consumer is then defined for all $(\nu, \xi) \in \mathcal{U} \times \mathcal{C}$ by

$$J_{\mathrm{A}}(\xi, \mathbb{P}^\nu) := \mathbb{E}^{\mathbb{P}^\nu}\Big[U_{\mathrm{A}}\Big(\xi + \int_0^T \big(f(X_s) - c(\nu_s)\big)\mathrm{d}s\Big)\Big], \text{ where } U_{\mathrm{A}}(x) := -\mathrm{e}^{-rx}, \tag{3.2}$$

for some constant risk aversion parameter $r > 0$. It is implicitly understood that the limiting case $r \searrow 0$ corresponds to a risk–neutral consumer.

The consumer's problem is

$$V_{\mathrm{A}}(\xi) := \sup_{\mathbb{P}^\nu \in \mathcal{P}} J_{\mathrm{A}}(\xi, \mathbb{P}^\nu), \tag{3.3}$$

i.e. maximising utility from deviating from the baseline electricity consumption subject to the cost of effort. A control $\mathbb{P}^{\widehat{\nu}} \in \mathcal{P}$ will be called optimal if $V^{\mathrm{A}}(\xi) = J_{\mathrm{A}}\big(\xi, \mathbb{P}^{\widehat{\nu}}\big)$. We denote by $\mathcal{P}^\star(\xi)$ the collection of all such optimal responses $\mathbb{P}^{\widehat{\nu}}$. We finally assume that the consumer has a reservation utility R

$$R = R_0 \mathrm{e}^{-r\pi}, \text{ where } R_0 := V_A(0). \tag{3.4}$$

Here, R_0 is the expected utility level that the Agent can achieve without contracting, and $-\mathrm{e}^{-r\pi}$ is the utility induced by the premium π. If π is positive, the consumer is asking for a premium over the utility without contract (because $R_0 < 0$). If π is negative, the consumers is willing to make some sacrifice to help the electricity system.

3.2 The producer

The producer (the Principal) provides electricity to the consumer, and thus faces the generation cost of the produced energy, and the cost induced by the variation of production. Her performance criterion is defined by

$$J_{\mathrm{P}}(\xi, \mathbb{P}^\nu) := \mathbb{E}^{\mathbb{P}^\nu}\Big[U\Big(-\xi - \int_0^T g(X_s)\mathrm{d}s - \frac{h}{2}\langle X\rangle_T\Big)\Big], \text{ with } U(x) := -\mathrm{e}^{-px}. \tag{3.5}$$

Here $p > 0$ is the constant absolute risk aversion parameter, g is a non–negative non–decreasing generation cost function, and h is a positive constant representing the *direct cost* induced by the quadratic variation

$\langle X \rangle$ of the consumption deviation. The higher the volatility of the consumption, the more costly it is for the producer to follow the load curve. Note that due to risk–aversion, the producer bears also an *indirect cost* of volatility.

An $\mathcal{F}_T$−measurable random variable ξ will therefore be called a *contract*, which we denote by $\xi \in \mathcal{C}$, if it satisfies the additional integrability property

$$\sup_{\mathbb{P}^\nu \in \mathcal{P}} \mathbb{E}^{\mathbb{P}^\nu}\big[e^{-rm\xi}\big] + \sup_{\mathbb{P}^\nu \in \mathcal{P}} \mathbb{E}^{\mathbb{P}^\nu}\big[e^{pm\xi}\big] < +\infty, \text{ for some } m > 1. \tag{3.6}$$

This integrability condition guarantees that the consumer criterion (3.2) and the principal one (3.5) are well-defined.

Throughout this paper, we shall consider the two following standard contracting problems.

- *First best contracting* corresponds to the benchmark situation where the producer has full power to impose a contract to the consumer and to dictate the effort the Agent effort

$$V^{\mathrm{FB}} := \sup_{(\xi,\mathbb{P}^\nu)\in\mathcal{C}\times\mathcal{P}} \big\{ J_{\mathrm{P}}(\xi,\mathbb{P}^\nu) : \; J_{\mathrm{A}}(\xi,\mathbb{P}^\nu) \geq R \big\}, \tag{3.7}$$

- *Second best contracting* allows the consumer to respond optimally to the producer offer. We follow the standard convention in the Principal–Agent literature in the case of multiple optimal responses in $\mathcal{P}^\star(\xi)$, that the consumer implements the optimal response that is the best for the producer. This leads to the second best contracting problem

$$V^{\mathrm{SB}} := \sup_{\xi\in\Xi} \sup_{\mathbb{P}^\nu\in\mathcal{P}^\star(\xi)} J_{\mathrm{P}}(\xi,\mathbb{P}^\nu), \text{ where } \Xi := \big\{ \xi \in \mathcal{C} : \; V_{\mathrm{A}}(\xi) \geq R \big\}, \tag{3.8}$$

  with the convention $\sup \emptyset = -\infty$, thus restricting the contracts that can be offered by the producer to those $\xi \in \mathcal{C}$ such that $\mathcal{P}^\star(\xi) \neq \emptyset$.

3.3 Consumer's optimal response and reservation utility

We collect here some calculations related to the consumer's optimal response which will be useful throughout the paper. Following Cvitanić et al. (2018) [13], we introduce the consumer's Hamiltonian.

$$H(z,\gamma) := H_{\mathrm{m}}(z) + H_{\mathrm{v}}(\gamma), \; z,\gamma \in \mathbb{R}, \tag{3.9}$$

where H_{m} and H_{v} are the components of the Hamiltonian corresponding to the instantaneous mean and the volatility, respectively

$$H_{\mathrm{m}}(z) := -\inf_{a\in A} \big\{ a\cdot \mathbf{1} z + c_1(a) \big\}, \text{ and } H_{\mathrm{v}}(\gamma) := -\frac{1}{2}\inf_{b\in B} \big\{ c_2(b) - \gamma|\sigma(b)|^2 \big\}. \tag{3.10}$$

Here, z represents the payment rate for a decrease of the consumption deviation and γ represents the payment rate for a decrease of the volatility of the consumption deviation. Both payments can be positive or negative. Given these payments, the consumer maximises the instantaneous rate of benefit given by the Hamiltonian to deduce the optimal response $\widehat{a}(z)$ on the drift and $\widehat{b}(\gamma)$ on the volatilities. The following result collects the closed–form expression of the last optimal responses. We denote $x^- := 0 \vee (-x)$, $x \in \mathbb{R}$.

Proposition 3.1. *The optimal response of the consumer to an instantaneous payment rate (z,γ) is*

$$\widehat{a}(z)_j := \mu_j(z^- \wedge A_{\max}), \quad \textit{and} \quad \widehat{b}_j(\gamma) := 1 \wedge (\lambda_j\gamma^-)^{-\frac{1}{1+\eta_j}} \vee \varepsilon, \; j = 1,\ldots,N,$$

so that, with $\bar{\mu} := \mu\cdot\mathbf{1}$, $\widehat{\sigma}(\gamma) := \sigma\big(\widehat{b}(\gamma)\big)$, $\widehat{c}_1(z) := c_1\big(\widehat{a}(z)\big)$, and $\widehat{c}_2(\gamma) := c_2\big(\widehat{b}(\gamma)\big)$,

$$H_{\mathrm{m}}(z) = \frac{1}{2}\bar{\mu}\big(z^- \wedge A_{\max}\big)^2 \quad \textit{and} \quad H_{\mathrm{v}}(\gamma) = -\frac{1}{2}\Big(\widehat{c}_2(\gamma) - \gamma\big|\widehat{\sigma}(\gamma)\big|^2\Big).$$

The z payment induces an effort of the consumer on all usages to reduce the average consumption deviation and this effort is proportional to its cost $1/\mu_i$. The γ payment induces an effort only on the usages whose cost $1/\lambda_j$ is lower than the payment. As a first use of the previous notations, we provide the following characterisation of the consumer's reservation utility. The proof is reported in the Appendix section A.1.

Proposition 3.2. (Consumer's reservation utility) *Assume that f is concave, non–decreasing, and Lipschitz–continuous. Then the following holds.*

(i) The consumer's reservation utility is concave in x_0, and is given by $R_0 = -\mathrm{e}^{-rE(0,x_0)}$, where the corresponding certainty equivalent E is a viscosity solution of the HJB *equation*

$$-\partial_t E = f + H_{\mathrm{v}}\big(E_{xx} - rE_x^2\big) \text{ on } [0,T)\times\mathbb{R}, \text{ and } E(T,x)=0,\ x\in\mathbb{R}, \tag{3.11}$$

with growth controlled by $|E(t,x)| \leq C(T-t)|x|$, for some constant $C>0$.

(ii) Assume that the PDE (3.11) *has a $C^{1,2}$ solution E with growth controlled by $|E(t,x)|\leq C(T-t)|x|$, for some constant $C>0$. Then the optimal effort of the consumer is defined by the feedback controls*

$$a^0 := 0, \text{ and } b_j^0 := \varepsilon \vee 1 \wedge \big(\lambda_j(E_{xx} - rE_x^2)\big)^{-\frac{1}{1+\eta_j}},\ j=1,\dots,d.$$

Note that, even without contracting, the consumer's optimal behaviour exhibits a positive responsiveness effort. The next result provides a closed–form expression for the reservation utility when f is linear.

Proposition 3.3. *Let $f(x)=\kappa\, x$, $x\in\mathbb{R}$, for some $\kappa\geq 0$. Then, the consumer's reservation utility is*

$$R_0 = -\mathrm{e}^{-r(\kappa x_0 T + E_0(T))}, \text{ where } E_0(T) := \int_0^T H_{\mathrm{v}}\big(-\gamma(t)\big)\mathrm{d}t, \text{ and } \gamma(t) := -r\kappa^2(T-t)^2.$$

The consumer's optimal effort on the drift and on each volatility usage are respectively

$$a^0 = 0, \text{ and } b_j^0(t) := \varepsilon \vee 1 \wedge \big(\lambda_j|\gamma(t)|\big)^{-\frac{1}{1+\eta_j}},\ j=1,\dots,d,$$

thus inducing an optimal distribution $\mathbb{P}^0$ under which the deviation process follows the dynamics $\mathrm{d}X_t = \widehat{\sigma}\big(b^0(t)\big)\cdot \mathrm{d}W_t$, for some $\mathbb{P}^0-$Brownian motion W.

Proof. By directly plugging the guess $E(t,x) = C(t)x + E_0(t)$ in the PDE (3.11), we obtain

$$C'(t)x + E_0'(t) + H_{\mathrm{v}}\big(-rC^2(t)\big) + \kappa x = 0, \text{ with } C(T) = E_0(T) = 0.$$

This entails $C(t) = \kappa(T-t)$ and $E_0(t) = \int_t^T H_{\mathrm{v}}\big(-rC^2(s)\big)\mathrm{d}s$, $0\leq t\leq T$. Finally the expression of the maximiser b^0 follows from Proposition 3.1. Since this smooth solution of the PDE has the appropriate linear growth, we conclude from Proposition 3.2 *(ii)* that it is indeed the value function inducing the reservation utility. □

Remark 3.1. In the setting of the last proposition, the consumer's certainty equivalent for $x_0=0$ is

$$\text{either } -\frac{1}{2}\int_0^T r\kappa^2(T-s)^2|\sigma|^2\mathrm{d}s, \quad \text{or} \quad -\frac{1}{2}\int_0^T \big(r\kappa^2(T-s)^2|\sigma(\gamma(s))|^2 + \widehat{c}_2(\gamma(s))\big)\mathrm{d}s,$$

depending on whether no effort is made or not. Hence, in both situations, the consumer bears a cost from the volatility and his optimal effort can reduce this cost but can not turn it into a benefit. This point is important as it implies that if the consumer does not require a premium π to enter in the contract, then setting the consumer at his reservation utility consists in taking away from him some cash.

4 First–best contracting

We provide here the solution of the first–best problem (3.7) corresponding to the case where the producer chooses both the contract ξ and the level of effort of the consumer ν, under the constraint that the consumer's satisfaction is above the reservation utility. Introducing a Lagrange multiplier $\ell \geq 0$ to penalise the participation constraint, and applying the classical Karush–Kuhn–Tucker method, we can formulate the producer's first–best problem as

$$V^{\mathrm{FB}} = \inf_{\ell \geq 0} \bigg\{ -\ell R + \sup_{(\xi, \mathbb{P}^\nu)} \mathbb{E}^{\mathbb{P}^\nu} \big[U\big(-\xi - \mathcal{K}_T^\nu \big) + \ell U_{\mathrm{A}} \big(\xi + G_T^\nu \big) \big] \bigg\}, \tag{4.1}$$

where $\mathcal{G}_T^\nu := \int_0^T g(X_s)\mathrm{d}s + \frac{h}{2}\langle X \rangle_T$ and $\mathcal{K}_T^\nu := \int_0^T (f(X_s) - c(\nu_s))\mathrm{d}s$. The first–order conditions in ξ are

$$-U'\big(-\xi_\ell - \mathcal{G}_T^\nu \big) + \ell U_{\mathrm{A}}'\big(\xi_\ell + \mathcal{K}_T^\nu \big) = 0.$$

In view of our specification of the utility functions, this provides the optimal contract payment for a given Lagrange multiplier ℓ

$$\xi_\ell := \frac{1}{p+r} \ln \left(\frac{r\ell}{p} \right) - \frac{p}{p+r} \mathcal{G}_T^\nu - \frac{r}{p+r} \mathcal{K}_T^\nu. \tag{4.2}$$

Substituting this expression in (4.1), we see that the Principal's first–best problem reduces to

$$V^{\mathrm{FB}} = \inf_{\ell \geq 0} \left\{ \ell \left(-R + \left(1 + \frac{r}{p} \right) \left(\frac{r\ell}{p} \right)^{\frac{-r}{r+p}} \bar{V} \right) \right\}, \text{ with } \bar{V} := \sup_{\mathbb{P}^\nu} \mathbb{E}^{\mathbb{P}^\nu} \Big[-\mathrm{e}^{-\rho \big(\int_0^T ((f-g)(X_t) - c(\nu_t))\mathrm{d}t - \frac{h}{2}\langle X \rangle_T \big)} \Big],$$

and $\frac{1}{\rho} := \frac{1}{r} + \frac{1}{p}$. Notice that $\bar{V}$ does not depend on the Lagrange multiplier ℓ. Then direct calculations lead to the optimal Lagrange multiplier and first–best value function

$$\ell^\star := \frac{p}{r} \left(\frac{\bar{V}}{R} \right)^{1+\frac{p}{r}}, \text{ so that } V^{\mathrm{FB}} = R \left(\frac{\bar{V}}{R} \right)^{1+\frac{p}{r}}. \tag{4.3}$$

This lead to the following proposition, whose proof is deferred to the Appendix section A.2.

Proposition 4.1. *Assume that $f - g$ is Lipschitz continuous.Then*

(i) $\bar{V} = -\mathrm{e}^{-\rho \bar{v}(0,x_0)}$, where $\bar{v}$ has growth $|\bar{v}(t,x)| \leq C(T-t)|x|$, for some constant $C > 0$, and is a viscosity solution of the PDE

$$-\partial_t \bar{v} = (f-g) + H_{\mathrm{m}}(\bar{v}_x) + H_{\mathrm{v}}\big(\bar{v}_{xx} - \rho \bar{v}_x^2 - h \big), \text{ on } [0,T) \times \mathbb{R}, \text{ and } \bar{v}(T,.) = 0, \tag{4.4}$$

so that, by (4.3), the first–best value function $V^{\mathrm{FB}} = U(\bar{v}(0,x_0) + \frac{1}{r}\log(-R))$.

(ii) If in addition $\bar{v}$ is smooth, the optimal efforts to induce a reduction of the consumption deviation and of its volatility are given by

$$a_{\mathrm{FB}}(t,X_t) := \widehat{a}\big(z_{\mathrm{FB}}(t,X_t) \big), \text{ and } b_{\mathrm{FB}}(t,X_t) := \widehat{b}\big(\gamma_{\mathrm{FB}}(t,X_t) \big), \ t \in [0,T], \tag{4.5}$$

where

$$z_{\mathrm{FB}}(t,x) := \bar{v}_x(t,x), \ \gamma_{\mathrm{FB}}(t,x) := \bar{v}_{xx}(t,x) - \rho \bar{v}_x^2(t,x) - h, \ (t,x) \in [0,T] \times \mathbb{R}.$$

(iii) Denoting $\nu_{\mathrm{FB}} := (a_{\mathrm{FB}}, b_{\mathrm{FB}})$, the optimal first–best contract can be written as

$$\xi_{\mathrm{FB}} = \frac{-\log(-R)}{r} - \frac{p}{p+r}\bar{v}(0,x_0) - \frac{r}{p+r} \int_0^T \big(f(X_t) - c(\nu_{\mathrm{FB}}(t,X_t)) \big)\mathrm{d}t - \frac{p}{p+r} \int_0^T g(X_t)\mathrm{d}t + \frac{h}{2}\langle X \rangle_T.$$

Remark 4.1. The optimal contract is a sum of the values and costs bore by the consumer and the producer weighted by their relative aversion. Producer's cost is taken back from the consumer and if the consumer reduces consumption, $f(X_t)$ is negative and thus, the consumer gets paid.

5 Second–best contracting

5.1 Revealing contracts

As the volatility induced by responsiveness effort is uniformly bounded above zero, and the level reduction effort is bounded, we may follow the general methodology of Cvitanić et al. (2018) [13], based on Sannikov (2008) [36]. Let $\mathcal{V}$ be the collection of all pair processes (Z,Γ) and constants $y_0\in\mathbb{R}$, inducing the subclass of contracts $\xi=Y_T^{y_0,Z,\Gamma}$, where

$$Y_t^{y_0,Z,\Gamma}:=y_0+\int_0^t Z_s\mathrm{d}X_s+\frac{1}{2}\int_0^t\big(\Gamma_s+rZ_s^2\big)\mathrm{d}\langle X\rangle_s-\int_0^t\big(H(Z_s,\Gamma_s)+f(X_s)\big)\mathrm{d}s,\ t\in[0,T]. \tag{5.1}$$

We recall from Cvitanić et al. (2018) that $U_A\big(Y_t^{y_0,Z,\Gamma}\big)$ represents the Agent's continuation utility so that $Y_t^{y_0,Z,\Gamma}$ is the time t value of the Agent's certainty equivalent. This contract is affine in the level of consumption deviation X and the corresponding quadratic variation $\langle X\rangle$, with linearity coefficients Z and Γ. The constant part $\int_0^T(H(Z_s,\Gamma_s)+f(X_s))\mathrm{d}s$ represents the certainty equivalent of the utility gain of the consumer that can be achieved by an optimal response to the contract, and is thus subtracted from the Principal's payment, in agreement with usual Principal–Agent moral hazard type of contract (see [30, Chapter 4]). Further, in the present setting, the risk aversion of the Agent implies that the infinitesimal payment $Z_t\mathrm{d}X_t$ must be compensated by the additional payment $\frac{1}{2}rZ_t^2\mathrm{d}\langle X\rangle_t$, so that, formally, the resulting impact of the payment $Z_t\mathrm{d}X_t$ on the Agent's expected utility is

$$-\exp\bigg(-r\bigg(Z_t\mathrm{d}X_t+\frac{1}{2}rZ_t^2\mathrm{d}\langle X\rangle_t\bigg)\bigg)+1\sim-1+r\bigg(Z_t\mathrm{d}X_t+\frac{1}{2}rZ_t^2\mathrm{d}\langle X\rangle_t\bigg)-\frac{1}{2}r^2Z_t^2\mathrm{d}\langle X\rangle_t+1\ \sim\ rZ_t\mathrm{d}X_t.$$

Under the optimal response of the consumer, the dynamics of the consumption deviation and the certainty equivalent of the consumer are given by

$$\begin{aligned}X_t^{Z,\Gamma}&:=x_0-\int_0^t\widehat{a}(Z_s)\cdot\mathbf{1}\mathrm{d}s+\int_0^t\widehat{\sigma}(\Gamma_s)\cdot\mathrm{d}W_s\\ Y_t^{Y_0,Z,\Gamma}&=Y_0+\int_0^t\Big(c\big(\widehat{a}(Z_s),\widehat{b}(\Gamma_s)\big)-f(X_s^{Z,\Gamma})+\frac{1}{2}rZ_s^2|\widehat{\sigma}(\Gamma_s)|^2\Big)\mathrm{d}s+\int_0^t Z_s\widehat{\sigma}(\Gamma_s)\cdot\mathrm{d}W_s,\end{aligned}$$

so that the average rate of payment consists in paying back the consumer her costs minus benefit $c-f$, and an additional compensation for the risk taken by the consumer for bearing the volatility of consumption deviation. Note that the average rate of payment can be positive or negative.

Remark 5.1. (i) Consider the zero premium case $\pi=0$, and assume that the producer proposes the contract ξ^0 defined by $y_0=-\frac{1}{r}\ln(-R_0)$, $Z=\Gamma\equiv 0$, i.e. $\xi^0=-\frac{1}{r}\ln(-R_0)-\int_0^T f(X_t^\nu)\mathrm{d}t$, as the Hamiltonian satisfies here $H_{\mathrm{v}}(0)\equiv 0$. Then, the optimal response of the consumer is obtained by solving the utility maximisation problem

$$V_A(\xi^0)=\sup_{\mathbb{P}^\nu\in\mathcal{P}}\mathbb{E}\Big[U_{\mathrm{A}}\Big(-\int_0^T c(\nu_t)\mathrm{d}t\Big)\Big].$$

As the cost function c is non–decreasing in the effort, at the optimum, the consumer makes no effort, neither on the drift, nor on the volatility. Comparing with Proposition 3.2, this shows that the absence of contract is different from the above contract ξ^0, with zero payment rates.

(ii) We may also examine the case where the producer offers a contract with payment $\xi=0$ to the consumer. This is achieved by choosing $Z_t=E_x(t,X_t)$ and $\Gamma_t=(E_{xx}-E_x^2)(t,X_t)$, where the certainty equivalent reservation E is defined in Proposition 3.2. From the point of view of the consumer, such a contract is clearly equivalent to the no contracting setting, and thus induces a positive effort on the volatility in the consumer's optimal response.

5.2 HJB characterisation of the producer's problem

By the main result of Cvitanić et al. (2018), we may reduce the Principal's problem to the optimisation over the class of contracts $Y_T^{y_0,Z,\Gamma}$, where $y_0 \geq -r^{-1}\log(-R)$ and $(Z,\Gamma)\in\mathcal{V}$. By the obvious monotonicity in y_0, this leads to the following standard stochastic control problem

$$V^{\rm SB} = \sup_{Z,\Gamma}\mathbb{E}\big[U\big(-L_T^{Z,\Gamma}\big)\big],\ \text{with}\ L_t^{Z,\Gamma} := Y_t^{Z,\Gamma} + \int_0^t g\big(X_s^{Z,\Gamma}\big)\mathrm{d}s + \frac{h}{2}\mathrm{d}\langle X^{Z,\Gamma}\rangle_s,\ t\in[0,T],$$

and starting point $L_0 = y_0 = -r^{-1}\log(-R)$. The state variable L represents the loss of the producer under the optimal response of the consumer, and is defined by the dynamics

$$\mathrm{d}L_t^{Z,\Gamma} = \frac{1}{2}\big(2(g-f)(X_t^{Z,\Gamma}) + \widehat{c}_1(Z_t) + f_0(rZ_t^2 + h,\Gamma_t)\big)\mathrm{d}t + Z_t\widehat{\sigma}(\Gamma_t)\cdot\mathrm{d}W_t,\ t\in[0,T).$$

where

$$f_0(q,\gamma) := q\big|\widehat{\sigma}(\gamma)\big|^2 + \widehat{c}_2(\gamma). \tag{5.2}$$

The function $f_0(q,\gamma)$ measures the total cost the producer incurs from the volatility, when the unit cost of volatility is q and the rate of payment for the volatility reduction is γ. The term $q|\widehat{\sigma}(\gamma)|^2$ is the instantaneous cost of volatility while the term $\widehat{c}_2(\gamma)$ is the cost of effort incurred by the consumer. This last cost will be paid by the producer, and enters thus in the evaluation of the cost of volatility. The producer aims at making the term $|\widehat{\sigma}(\gamma)|^2$ as small as possible. To achieve this objective, a sufficiently large γ should be paid to reduce $|\widehat{\sigma}(\gamma)|^2$, but this can be done only at the expense of an increasing cost $\widehat{c}_2(\gamma)$.

Lemma 5.1. *Let $F_0(q) := \inf_{\gamma\leq 0} f_0(q,\gamma)$. Then $F_0(q) = f_0(q,-q) = -2H_{\rm v}(-q)$ is non–decreasing.*

Proof: See Appendix A.3. □

The value function of the second–best problem can be characterised as follows, see Section A.4.

Proposition 5.1 (Second-best contract)**.** *Assume that $f-g$ is Lipschitz continuous. Then*

(i) $V^{\rm SB} = -\mathrm{e}^{-p(v(0,x_0)-L_0)}$ where v has growth $|v(t,x)|\leq C(T-t)|x|$, for some constant $C>0$, and is a viscosity solution of the PDE

$$\begin{cases} -\partial_t v = f - g + \frac{1}{2}\bar{\mu}\,v_x^2 - \frac{1}{2}\inf_{z\in\mathbb{R}}\big\{F_0\big(q(v_x,v_{xx},z)\big) + \bar{\mu}\big((z^- + v_x)^2 + \eta_A(v_x,z)\big)\big\},\ \text{on}\ [0,T)\times\mathbb{R}, \\ v(T,.) = 0, \end{cases} \tag{5.3}$$

with $q(v_x,v_{xx},z) := h - v_{xx} + rz^2 + p(z-v_x)^2$, and $\eta_A(v_x,z) := (v_x + (z^- - A)^+)^2 - v_x^2 \longrightarrow 0$, as $A\nearrow\infty$.

(ii) If in addition v is smooth, the optimal payment rate $\gamma_{\rm SB}$ to incentivise the agent responsiveness is

$$\gamma_{\rm SB}(t,X_t) := -h + v_{xx}(t,X_t) - rz_{\rm SB}^2(t,X_t) - p\big(z_{\rm SB}(t,X_t) - v_x(t,X_t)\big)^2,\ t\in[0,T], \tag{5.4}$$

and the optimal payment rate for the consumption deviation reduction is the minimiser $z_{\rm SB}$ in (5.3), satisfying for large A:

$$z_{\rm SB}\in\left(v_x, \frac{p}{r+p}v_x\right),\ \text{when}\ v_x\leq 0,\ \text{and}\ z_{\rm SB} = \frac{p}{r+p}v_x,\ \text{when}\ v_x\geq 0.$$

(iii) The second–best optimal contract is given by

$$\xi_{\rm SB} := \frac{-\log(-R)}{r} + \int_0^T z_{\rm SB}(t,X_t)\mathrm{d}X_t + \frac{1}{2}(\gamma_{\rm SB} + rz_{\rm SB}^2)(t,X_t)\mathrm{d}\langle X\rangle_t - \big(H(z_{\rm SB},\gamma_{\rm SB}) + f\big)(t,X_t)\mathrm{d}t.$$

Remark 5.2. (i) Consider the case of a risk–neutral consumer $r = 0$. As F_0 is non–decreasing by Lemma 5.1, we see that the minimum in the PDE (5.3) is attained at $z_{\rm SB} = v_x$, thus reducing the PDE (5.3) to

$$-\partial_t v - (f - g) - \frac{1}{2}\bar{\mu}\left(v_x^-\right)^2 = -\frac{1}{2}F_0(h - v_{xx}) = H_{\rm v}(v_{xx} - h),$$

where the last equality follows from Lemma 5.1. Notice that this is the same PDE as the first–best characterisation given in Proposition 4.1 (i), since $\rho = 0$ in the present setting.

In particular, the producer's value function is independent of her risk–aversion parameter p. The optimal payment rates for the effort on the drift and the volatility are given by $z_{\rm SB} = v_x$ and $\gamma_{\rm SB} = -h + v_{xx}$, so that the resulting optimal contract is also independent of the producer's risk–aversion p. This is consistent with the findings of Hölmstrom and Milgrom [23] in the context of a risk–neutral agent, where the optimal effort of the Agent is independent of the Principal risk aversion parameter.

Remark 5.3. (i) From Proposition 3.1, a positive payment z induces no effort of the consumer on his average deviation, as $\widehat{a}(z) = \mu z^- \wedge A_{\max}$. However when $v_x > 0$, it follows from Proposition 5.1 that the optimal payment rate $z_{\rm SB} = \frac{p}{r+p} v_x$ is also positive, and not zero, as one could expect. In other words, the producer is paying the consumer when the consumption deviation increases. While this finding seems to be in contradiction with the producer's objective, it is explained as follows. Observe that the optimal rate of payment $z_{\rm SB}$, as the minimiser of $q(v_x, v_{xx}, z)$, balances two effects

- on one hand, the producer compensates the positive efforts of the consumers on the drift reduction;
- on the other hand, the producer aims at reducing the squared distance $|z - v_x|^2$ representing her indirect cost of volatility, as her continuation utility $-e^{-p(v(t,X_t) - L_t^{Z,\Gamma})}$ is affected by the volatility $v_x(t, X_t) - Z_t$ of the difference $v(t, X_t) - L_t^{Z,\Gamma}$ by $p(v_x(t, X_t) - Z_t)^2$; this is a direct consequence of Itô's formula.

(ii) Notice that the case $v_x \geq 0$ cannot be excluded. A positive marginal value of this certainty equivalent means that the higher the initial deviation, the higher the utility of the producer. Using the expression of the objective function of the producer in (3.5) in our formula for the optimal contract in (5.1), one can check that the monotonicity of the value function of the producer w.r.t. the initial condition x depends only on $\int_0^T (f - g)(X_t)\mathrm{d}t$. Thus, if the consumer values more the energy than it is costly for the producer to generate it, the value function of the producer is increasing in x. This remark explains why the producer will not encourage the consumer to reduce his consumption in this situation; see Section 6 for more details in the case of linear energy valuation and generation cost.
(iii) If the consumer asks for no premium π to enter in the contract, the producer may have an interest in signing the contract even if there is no direct volatility cost $h = 0$. Indeed, the producer's risk–aversion induces a volatility cost depending on the risk aversion parameter p, and induces a gain from signing the contract even if $h = 0$.

6 Linear case

We consider in this section the case where

$$(f - g)(x) := \delta x, \; x \in \mathbb{R}. \tag{6.1}$$

for some constant parameter δ, called hereafter *energy value discrepancy*. The case $\delta \geq 0$ corresponds to the setting where an increase in consumption provides more utility to the consumer than the additional generation cost induced to the producer. It corresponds to off–peak hours. Similarly, negative δ means that an increase of consumption induces more cost for the producer than the benefit gained by the consumer. It corresponds to peak–load hours.

We directly consider the limiting case

$$\epsilon = 0 \quad \text{and} \quad A_{\max} = 0,$$

although our explicit derivation of the value function of the producer can also be obtained for fixed ϵ and $A_{\max}$.

6.1 Producer's benefit from the second–best contract

Similar to the proof of Proposition 3.3, we derive a closed–form solution for the PDE (5.3) of the form

$$v(t,x) \;=\; A(t)\,x + B(t), \tag{6.2}$$

for some appropriate functions A and B. To determine the functions A and B, we plug this guess into the PDE (5.3), and obtain the following restrictions:

$$\begin{cases} -A'(t) \;=\; \delta, \quad A(T) = 0, \\ -B'(t) \;=\; m(t) \;:=\; \frac{1}{2}\bar{\mu}A^2(t) - \frac{1}{2}\inf_{z\in\mathbb{R}} \big\{ F_0\big(h + rz^2 + p(z - A(t))^2\big) + \bar{\mu}(z^- + A(t))^2 \big\}, \; B(T) = 0. \end{cases}$$

This provides

$$A(t) = \delta(T - t), \quad \text{and} \quad B(t) = \int_t^T m(s)ds, \; t \in [0,T], \tag{6.3}$$

so that the sign of v_x is determined by that of δ. The derivation of the function B will be continued in the next subsections. In particular, it turns out that the consumer behavior is crucially driven by the sign of the parameter δ.

Notice that, the solution (6.2)-(6.3) derived by this guess has the appropriate linear growth, as required in Proposition 5.1. We may then conclude that it coincides with the certainty equivalent function v which induces the second–best producer's value function.

Proposition 6.1. *Let the energy value discrepancy be linear as in* (6.1)*. Then:*
(i) the producer's second–best value function is given by $V^{\rm SB} = U\big(v(0,X_0) - L_0\big)$*, with certainty equivalent function* v *characterized by* (6.2)-(6.3)*. The optimal payment rates are the deterministic functions*

$$z_{\rm SB} = \operatorname{Arg}\min_{z\in\mathbb{R}} \big\{ F_0\big(h + rz^2 + p(z-A)^2\big) + \bar{\mu}(z^- + A)^2 \big\}, \; \textit{and} \; \gamma_{\rm SB} = -\big(h + rz_{\rm SB}^2 + p(z_{\rm SB} - A)^2\big).$$

(ii) the second–best optimal contract is given by

$$\xi_{\rm SB} = \frac{-\log(-R)}{r} + z_{\rm SB}(0)X_0 - \int_0^T H(z_{\rm SB},\gamma_{\rm SB})(t)dt + \frac{1}{2}\int_0^T (\gamma_{\rm SB} + rz_{\rm SB}^2)d\langle X\rangle_t - \int_0^T (\kappa + z'_{\rm SB}(t))X_t dt.$$

Proof. The claim follows from the fact that $V^{\rm SB} = -e^{-p[v(0,X_0) - L_0]}$, with $L_0 = -r^{-1}\log(-R)$, together with the calculation preceding the statement. The form of the contract is obtained by applying integration by part to the term $\int_0^T z_{\rm SB}(t)dX_t$ of the general form of the contract (5.1). □

Remark 6.1. In the case of linear energy value, the optimal contract shares the form of real–life demand response contracts. It is the sum of a constant term

$$\frac{-\log(-R)}{r} + z_{\rm SB}(0)X_0 - \int_0^T H(z_{\rm SB},\gamma_{\rm SB})(t)dt$$

representing a premium paid at the enrollment, and a term proportional to the consumption but depending on the realised effort of the consumer

$$\frac{1}{2}\int_0^T (\gamma_{\mathrm{SB}} + r z_{\mathrm{SB}}^2) d\langle X\rangle_t - \int_0^T (\kappa + z'_{\mathrm{SB}}(t)) X_t dt.$$

However, in our case, the payment or penalty applies to the consumption deviation and not the total consumption. Nevertheless, a payment for the deterministic part of the consumption could be added up to the constant term. Further, we note that the constant term depends on the duration of the effort T. Thus, the different initial payment should be paid if the producer wants to implement a response for a price event of three, six or 12 hours.

In the case where the energy value discrepancy is positive it is possible to obtain more explicit result for Proposition 6.1.

Proposition 6.2. *Suppose that the energy has more value for the consumer than for the producer, i.e.* $(f-g)' = \delta \geq 0$. *Then:*
(i) *the optimal payments rate are deterministic functions of time given by*

$$z_{\mathrm{SB}}(t) = \frac{p}{r+p}\delta(T-t), \quad \textit{and} \quad \gamma_{\mathrm{SB}}(t) = h + \frac{rp}{r+p}\,\delta^2(T-t)^2$$

inducing the second-best optimal contract

$$\xi_{\mathrm{SB}} = \frac{-\log(-R)}{r} + \frac{p\delta T}{r+p}X_0 - \int_0^T H(0,\gamma_{\mathrm{SB}}(t))dt + \frac{1}{2}\int_0^T \big(\gamma_{\mathrm{SB}}(t) + r z_{\mathrm{SB}}(t)^2\big) d\langle X\rangle_t - \int_0^T (\kappa + \frac{p}{r+p}\delta) X_t dt. \tag{6.4}$$

(ii) *the consumer's optimal effort on the drift and the volatility of the consumption deviation is*

$$\widehat{a}\big(z_{\mathrm{SB}}\big) = 0, \ \textit{and}\ \widehat{b}_j\big(\gamma_{\mathrm{SB}}\big) = 1 \wedge \Big(\lambda_j \gamma_{\mathrm{SB}}(t)\Big)^{-\frac{1}{1+\eta_j}},$$

so that the optimal response of the consumer induces the optimal probability distribution $\mathbb{P}^{\mathrm{SB}}$ *such that* $dX_t = \widehat{\sigma}(\gamma_{\mathrm{SB}}) \cdot dW_t$, *for some* $\mathbb{P}^{\mathrm{SB}}-$*Brownian motion* W.

Proof. (ii) Applying Proposition 6.2 and Proposition 5.1 (iii) to the case where $\delta \geq 0$ gives directly that $z_{\mathrm{SB}} = \frac{p}{r+p} v_x = \frac{p}{r+p}\delta(T-t)$ and thus γ_{SB} follows. Further, applying Proposition 5.1 (iv) together with the expression of z_{SB} gives directly ξ_{SB}. (iii) The application of Proposition 3.1 gives the result. Reassembling the former results together with Proposition 6.2 (i) and our guess, gives the expression of V^{SB} in (i). $\square$

Remark 6.2. (i) Consider the special case where the producer and the consumer agree on the energy value, i.e. $\delta = 0$. The payment rates to reduce volatility is a constant and does not depend on the risk–aversion of the producer and the consumer. It is just a matter of direct cost of volatility for the producer h and volatility cost reduction for the consumer $1/\bar{\lambda}$. The consumer makes effort only on those usages whose marginal cost of effort $1/\lambda_j$ is lower than the producer's marginal cost of volatility h. For each usage, the volatility is a constant, equal to σ_j if $h < 1/\lambda_j$, or to $\sigma_j(\lambda_j h)^{-1/(2(1+\eta_j))}$ otherwise.

(ii) If in addition the producer incurs no volatility cost $h = 0$, then the optimal payment γ_{SB} induces no effort from the consumer. Thus, the volatility of the consumption deviation is larger than when there is no contract and the consumer makes effort to reduce his own induced cost of volatility. One can expect a benefit for the producer from trading volatility with the consumer — and we conduct the analysis of this benefit after this remark. But, surprisingly, by transferring the volatility risk from the consumer to the producer, the optimal contract can lead to an increase of volatility. From a risk–sharing point of view, this means that optimal contracting allows the system to bear more risk.

(iii) When $\delta > 0$, the producer induces a time–decreasing effort on the volatilities. Thus, depending on the relative costs of energy δ, relative risk–aversion r and p and duration of the contract T, it may happen that at some point during the execution of the contract, the producer stops requiring an effort from the consumer. Further, the producer requires no effort on the drift but compensates the consumer for the volatility costs induced by increasing or decreasing consumption. Indeed, we see below that there is no reason for the producer to require that the consumer reduces the consumption as the generation cost is less than the producer benefit.

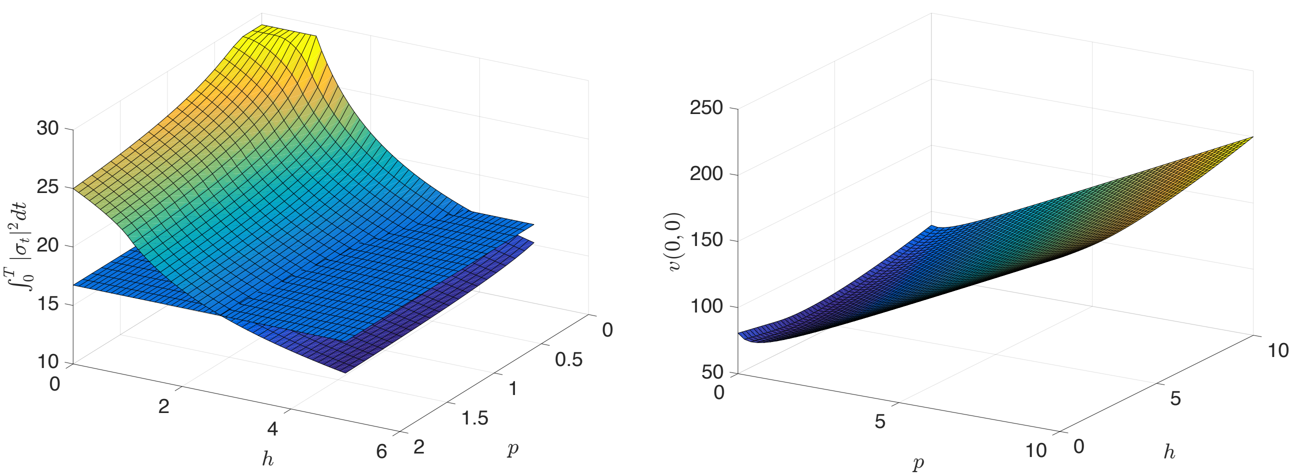

Figure 2: (Left) Total volatility of consumption deviation under optimal contract as a function of the direct volatility cost h and the risk-aversion parameter p of the consumer compared to the total volatility without contract (flat surface). (Right) Certainty equivalent of the producer with contract and without contract as a function of the direct volatility cost h and the risk-aversion parameter p. parameters value: two usages, $T = 1$, $\mu = (2, 5)$, $\sigma = (5, 2)$, $\lambda = (0.5, 0.1)$, $\eta = (1, 1)$, $\kappa = 5$, $\delta = 3$.

Figure 2 illustrates the variation of the observed total volatility of consumption deviation and the benefit from the contract as a function of the direct marginal cost of volatility h and the risk-aversion parameter of the producer p when $\delta > 0$. For a risk neutral producer with zero marginal direct cost of volatility, the producer requires no effort from the consumer and the volatility is equal to the nominal volatility. For a risk neutral producer with increasing direct marginal cost of volatility, we observe that volatility remains constant until the value of h gets higher than a certain threshold, namely the lower cost of effort for volatility reduction of the consumer. Then, the producer starts to require an effort from the consumer. Then, the higher the value of h, the lower the observed volatility until an effort is required that reduces the volatility below its value before contracting. If we fix the value of h and make the producer increasingly risk–averse, we also observe a progressive yet slower reduction of volatility because the indirect cost of volatility is much lower than its direct cost. Further, we observe that the gain from the contract is a non–decreasing function of p but a non-monotonic function of h. Indeed, even if the benefit from the contract is always positive (the risk-premium π is set to zero), the producer who has a small direct marginal cost does not induce any effort from the consumer and thus, cannot reap any increased benefit from the behaviour of the consumer.

Remark 6.3. When the energy has less value for the consumer than for the producer, $\delta < 0$, there is a potential for the producer to exchange a costly energy generation against some payment for consumption reduction. Unfortunately, there is no closed–form expression for $z_{\rm SB}$ in general, though. It is possible to derive an explicit expression of the optimal payment for the drift in the case where

$$h + r\delta^2 T^2 \quad \leq \quad \frac{1}{\bar{\lambda}}, \tag{6.5}$$

i.e. when the marginal cost of volatility augmented by the risk premium is still lower than the lowest marginal cost of volatility reduction for the consumer. In this case, the optimal payment rates are given by

$$z_{\rm SB}(t) = \Lambda\delta(T-t), \quad \text{and} \quad \gamma_{\rm SB}(t) = -\frac{1}{\bar{\lambda}}, \quad \text{with} \quad \Lambda := \frac{p|\sigma|^2 + \bar{\mu}}{(p+r)|\sigma|^2 + \bar{\mu}}.$$

The optimal response of the consumer under the optimal contract induce the optimal probability distribution $\mathbb{P}^{\rm SB}$ so that $dX_t = \bar{\mu} z_{\rm SB}(t)dt + \sigma \cdot dW_t$, for some $\mathbb{P}^{\rm SB}-$Brownian motion W. The optimal contract is

$$\xi_{\rm SB} = \frac{-\log(-R)}{r} + \Lambda\delta X_0 + \frac{1}{2}\int_0^T (\gamma_{\rm SB}(t) + r z_{\rm SB}^2(t))d\langle X\rangle_t - \int_0^T H_{\rm m}(z_{\rm SB}(t), \gamma_{\rm SB}(t))dt - \int_0^T (\kappa + \Lambda\delta)X_t dt.$$

In the present setting, the producer induces no effort of the consumer on the volatilities. Thus, when energy has less value for the consumer but the marginal cost of effort to reduce volatility is higher than the marginal cost for the producer, the producer only induces an effort on the drift. On the other hand, when there is at least one usage such that $\frac{1}{\lambda_j} \leq h + r\delta^2 T^2$, the producer also induces an effort on the volatilities of usage whose marginal cost of reduction is lower than her marginal cost of volatility. In addition, the absolute value of the optimal payment $z_{\rm SB}$ is decreasing with the volatility. This is consistent with the Principal–Agent literature, the more the volatility obfuscates the result of the effort of the Agent, the less the Principal is prone to pay for them. Also, notice that the consumer's optimal effort in the drift, $\bar{\mu}(z_{\rm SB}(t))^-$, is actually decreasing with time.

6.2 First–best versus second–best

In the present linear setting, we may also solve the PDE (4.4) in closed form, and deduce the optimal effort of the consumer.

Proposition 6.3. *Under (6.1), the consumer's optimal effort is given by*

$$a_{\rm FB}(t) := \mu\delta^-(T-t), \quad \textit{and} \quad b_{\rm FB}(t) := 1 \wedge \Big(\lambda_j(h + \rho\,\delta^2(T-t)^2)\Big)^{-\frac{1}{1+\eta_j}},$$

thus inducing an optimal distribution $\mathbb{P}^{\rm FB}$ under which the deviation process follows the dynamics $dX_t = -\mu\delta^-(T-t)\cdot \mathbf{1}dt + \sigma\big(b_{\rm FB}(t)\big)\cdot dW_t$, for some $\mathbb{P}^{\rm FB}-$Brownian motion W.

Define the informational rent as $I := -\frac{1}{p}\log\big(\frac{V^{\rm FB}}{V^{\rm P}}\big)$.

Proposition 6.4. *If $\delta \geq 0$, then there is no informational rent, $I = 0$ and $\xi_{\rm SB} = \xi_{\rm FB}$. When $\delta \leq 0$ and $h + r\delta^2 T^2 \leq \frac{1}{\bar{\lambda}}$, the informational rent is*

$$I \quad = \quad \frac{\delta^2 T^3 r^2}{6(p+r)} \, \frac{1}{\frac{1}{|\sigma|^2} + \frac{p+r}{\bar{\mu}}}.$$

Proof. From Proposition 4.1, we have $V^{\rm FB} = U(\bar{v}(0,X_0) + \frac{1}{r}\log(-R))$. When $(f-g)(x) = \delta x$, we have

$$\bar{v}(0,X_0) = \delta T X_0 - \frac{1}{2}\int_0^T F_0(-\gamma_{\rm FB}(t))dt,$$

because $z_{\rm FB}(t) \geq 0$ and thus, $\widehat{c}_1(z_{\rm FB}(t)) = 0$. Further, recall from Proposition 6.1 that when $\delta \geq 0$, we have $V^{\rm SB} = U(v(0,X_0) + \frac{1}{r}\log(-R))$ with

$$v(0,X_0) = \delta T X_0 - \frac{1}{2}\int_0^T F_0(-\gamma_{\rm SB}(t))dt.$$

In this case, we also have $\gamma_{\rm SB} = \gamma_{\rm FB}$. Thus, the certainty equivalent of the value function in the first–best and in the second–best are equal, and $I = 0$. Further, the equality of the certainty equivalent of the first–best and the second–best implies that the payments $\xi_{\rm FB}$ and $\xi_{\rm SB}$ are equal because the actions of the consumer are the same in both cases.

When $\delta < 0$ and $h + r\delta^2 T^2 \leq \frac{1}{\lambda}$, we know that $z_{\rm SB}(t) = \Lambda\delta(T-t)$, so that

$$\inf_{z\in\mathbb{R}} \Big\{ F_0\big(h + rz^2 + p(z - A(t))^2\big) + \bar{\mu}(z^- + A(t))^2 \Big\} = \big(h + r\Lambda\delta^2(T-t)^2\big)|\sigma|^2,$$

and therefore

$$\psi(t) = \frac{1}{2}\int_t^T \bar{\mu}\delta^2(T-t)^2 dt - \frac{1}{2}\int_0^T \Big(h + r\Lambda\delta^2(T-t)^2\Big)|\sigma|^2 dt.$$

Hence, in this case

$$\begin{aligned}\frac{\log\big(-V^{\rm SB}\big)}{p} &= L_0 - \delta T x_0 - \psi(0)\\ &= \pi + \frac{1}{2}\int_0^T \Big(\gamma_s\big|\widehat{\sigma}\big(\gamma_s\big)\big|^2 - \widehat{c}_2\big(\gamma_s\big) - \bar{\mu}\delta^2(T-s)^2\Big)ds + \frac{1}{2}\int_0^T \Big(h + r\Lambda\delta^2(T-t)^2\Big)|\sigma|^2 dt.\end{aligned}$$

In addition, in this setting, $c(\nu^{\rm FB}) = c_1(a_{\rm FB})$ and $-\gamma_{\rm FB} = h + \rho\delta^2(T-t)^2 \leq \frac{1}{\lambda}$ because $\rho < r$, and thus $\widehat{b}_j(\gamma_{\rm FB}) = 1$, and we have $c_1(a_{\rm FB}(t)) = \frac{1}{2}\bar{\mu}\delta^2(T-t)^2$. Thus we get

$$I = \frac{1}{2}\int_0^T \gamma_{\rm FB}(t)\big|\widehat{\sigma}\big(b_{\rm FB}(t)\big)\big|^2 dt + \frac{1}{2}\int_0^T \big(h + r\Lambda\delta^2(T-t)^2\big)|\sigma|^2 dt = \frac{1}{2}|\sigma|^2\big(r\Lambda - \rho\big)\int_0^T \delta^2(T-t)^2 dt,$$

where we used the fact that $-\gamma_{\rm FB} = h + \rho\delta^2(T-t)^2$. The required expression follows. □

If the energy valuation discrepancy is positive, the only value that the consumer can keep from the producer is the value from the initial condition of consumption deviation. In this case, in terms of volatility reduction, the optimal contract implements the same efforts as would do the social planner.

7 Empirical results

The purpose of this section is to provide an estimate of the potential gains from the implementation of a responsiveness incentive mechanism, both in terms of volatility reduction and of benefits for the producer. We first calibrate our model with linear energy valuation and energy generation cost on a publicly available data set of a demand response experiment. We next estimate a conservative value for the gains from responsiveness control, and we examine the robustness of the linearity assumption on energy valuation and energy cost.

7.1 Data

We use the publicly available consumption data provided by the Low Carbon London Project of demand-side response (DSR) trial performed in 2012-2013. The data is publicly available at London DataStore website (https://data.london.gov.uk) under the section *Smart Meter Energy Consumption Data in London Households*. The demand response trial is extensively described in a series of reports amongst which the reports Tindemans et al. (2014) and Schofield (2014) [39, 37] are the most relevant for our study. This experiment was conducted at the initiative of the UK energy regulator (Ofgem) in partnership with many players and academic institutions amongst which two have to be cited for our study, Imperial College who treated the data of the experiment and EDF Energy who acted as the energy provider and enroller for the consumers. The data consists in a set of 5,567 London households whose consumption have been measured at an half hourly time-step on the period from February, 2011 to February, 2014. For the dynamic Time-of-Use (dToU) tariff trial, the population was divided in two groups. One group of approximately 1,117 households were enrolled by EDF Energy in the dToU tariff while the remaining 4,500 households were not subject to this dynamic tariff.

The dToU was applied during the year 2013 (January, 1st to December, 31st). Tariffs were sent to the households on a day-ahead basis using a Home Display or a text message to the customer mobile phone. Prices had three levels: High (67.20 p/kWh), Normal (11.76 p/kWh) and Low (3.99 p/kWh). Standard tariff is made of a flat tariff of 14.228 p/kWh.

The precise description of the dToU trial performed in 2013 is given in Tindemans et al. (2014) [39, chap. 3]. The total number of events (High and Low) were 93 to deal with supply events (shortage of generation) and 21 for distribution network events. In our study, we are only interested in the High price events. They were 69 such events of High prices (45 for supply reasons and 24 for network reasons). The duration of an event could be 3, 6, 12 and 24 hours. The Low Carbon London Demand–Side Response Trial was designed to be as close as possible to a random trial experiment, while accounting for the operational constraints related to the enrollment of a large set of customers within the portfolio of given UK utility (EDF Energy). The events were randomly placed over the trial period while targeting the highest peaks of demand in the year.

Out of this dataset, we eliminated all consumers for which data were not complete or exhibit outliers. The resulting sample consists in 880 consumers in the control group and 250 consumers in the dToU group.

7.2 Calibration

In order to assess the benefits from the control of the responsiveness, we provide the optimal contract and the value function of the producer in the case where incentives are limited to payments for efforts on the drifts and the energy value is linear. We consider the problem (3.8) of the producer where the contracts (5.1) are limited to the pairs $(Z, 0)$. We denote by $V^{\mathrm{SB_m}}$ the value of the producer's problem in this case.

Proposition 7.1 (Second best uncontrolled responsiveness). *Assume that $f - g = \delta x$. Then, $V^{\mathrm{SB_m}} = U\big(w(0, X_0) - L_0\big)$ where w has growth $|w(t,x)| \leq C(T-t)|x|$, for some constant $C > 0$, and is a viscosity solution of the* PDE

$$-\partial_t w = (f-g) + \frac{1}{2}\bar{\mu}\, w_x^2 - \frac{1}{2}\inf_{z\in\mathbb{R}}\big\{q(w_x, w_{xx}, z)|\sigma|^2 + \bar{\mu}(z^- + w_x)^2\big\}, \text{ on } [0,T)\times\mathbb{R}, \text{ and } w(T,.) = 0, \quad (7.1)$$

with $q(w_x, w_{xx}, z) = h - w_{xx} + rz^2 + p(z - w_x)^2$, and the second–best optimal contract is given by

$$\xi^0_{\mathrm{SB}} = \frac{-\log(-R)}{r} + \Lambda\delta X_0 + \frac{1}{2}\int_0^T r z^2_{\mathrm{SB}}(t)|\sigma|^2 dt - \int_0^T H_{\mathrm{m}}(z_{\mathrm{SB}}(t))dt - \int_0^T (\kappa - \Lambda\delta)X_t dt$$

where $z_{\mathrm{SB}} = \Lambda\delta(T-t)$ *with* $\Lambda := \frac{p|\sigma|^2 + \bar{\mu}\mathbf{1}_{\{w_x<0\}}}{(p+r)|\sigma|^2 + \bar{\mu}\mathbf{1}_{\{w_x<0\}}}$.

Proof. When $\Gamma \equiv 0$, we have $F_0(q) = q|\sigma|^2$, and the PDE of Proposition 5.1 (i) reduces thus to (7.1). The minimizer is obtained directly by writing first order conditions, and the optimal contract follows. The form of the contract is obtained by applying integration by part to the term $\int_0^T z_{\mathrm{SB}}(t)dX_t$ of the general form of the contract (5.1). □

As mentioned in Section 6.1, the optimal contract is the sum of a constant term plus a term proportional to the consumption deviation. Thus, the optimal contract has the same form as the LCL pricing trial contract: a fixed premium to get enrolled plus a term proportional to the consumption. This provides the rationality for the calibration of the optimal contract without responsiveness control to the data of the LCL pricing trial. Thus, our strategy to calibrate our model is to used Proprosition 7.1 to answer to the question: what should be the parameters value of the consumer's behaviour model μ that would lead to the observed consumption reduction of the LCL pricing trial? Regarding the parameters of the consumer's cost of effort for responsiveness λ and η, no data allows to estimate them as no pricing trial were even done with the objective in mind to reduce the variance of responses. Thus, we make the following thought experiment: given that the consumer makes an effort to increase his responsiveness, what could be the highest volatility reduction cost that the consumer could bear from responding to the producer incentives? For a fixed λ, we compute numerically the total cost of volatility reduction $\int_0^T \frac{1}{2}\widehat{c}_2(\gamma_t)dt$ and then, we look for the λ that maximizes this cost. Then, we compare this cost to the cost the consumer already bears when reducing his average consumption.

We report in Table 2 below the calibration of our model, and we justify our choices in the rest of this section.

Duration of the price event T. In the LCL pricing trial, there were 69 High Price events for a total of 778 half-hours. The events could last 3, 6, 12 or 24 hours. Only one exceptional event lasted a full day (24 hours). Removing this outlier, we find an average duration of price event of 5.44 hours. We set $T = 5.5$ hours.

Energy value parameters κ and δ. For the consumer, we set $\kappa = 11.76$ pence/kWh, which is the price the consumer enrolled in the dToU group pays in normal situation. The surplus generation cost parameter δ is negative so that the producer has an interest for consumption reduction. We set the value of δ by considering that the High Price value (67.2 pence/kWh) of the pricing trial corresponds to the energy value for the producer. Hence, $\delta = 11.76 - 67.2 = -55.44$ pence/kWh.

Nominal volatility σ. We take as nominal volatility σ the estimate of the volatility of consumption deviation of the control group given in Table 1. Namely, we set $\sigma = 0.087$ W/h$^{1/2}$.

Producer's risk-aversion p. Let S denote the spot-price of electricity for a given hour and F the forward price quoted the day before, one has $\mathbb{E}[e^{-pS}] \approx e^{-p(\mathbb{E}[S] - \frac{1}{2}p\sigma_S^2)}$, and by equating the certainty equivalent with the forward price, we obtain the risk-premium RP $:= F - \mathbb{E}[S] = \frac{1}{2}p\sigma_S^2$. The risk-premium electricity utilities are ready to pay to avoid the day-ahead spot price risk has been extensively analyzed and estimated in the financial economics literature. Bessembinder and Lemon (2002) [5] followed by Longstaff and Wang [32], Benth et al. [4] and Viehmann (2011) [42] estimated the relation between the risk premia on each hour of delivery and the variance of the spot price on this hour. They find consistent and convergent estimation both on the sign of the risk premia (negative for off-peak hours and positive for peak hours). If one focuses on the peakest hour of the day (typically 7 or 8pm), the former authors find that dependance of the risk-premium with respect to the variance of the spot price is 0.31 for Viehmann (2011) (Table 5, hour 20), which makes $p = 0.62$; Benth et al. (2008) estimates that p is no lower than 0.421 (page 14); Longstaff and Wang (2004) find a dependence of the risk-premium to the variance of the spot price of 0.29 (page 1895, Table VI, hour 20), which makes $p = 0.58$. Bessembinder and Lemon (2002) estimates risk-premia not for day-ahead spot price risk but for monthly prices, which is less relevant in our context. Thus, we take as a nominal value for the risk-aversion parameter of the producer $p = 0.6$ per

pound.
Consumer's risk-aversion r. There is a large and not necessarily consensual economic literature on the relevant estimation of consumer's risk-aversion parameters, in particular when using CARA utility function (see Gollier (2001) monography). Nevertheless, in the context of the LCL Pricing trial, the consumers were facing a small variation of their electricity bill which is itself a fraction of their expenses, making the approximation of independence of decision with respect to wealth sustainable. Further, it is possible to provide an estimate of the risk-aversion parameter r of the population who accepted to enroll in the dynamic ToU. Indeed, the enrolled consumers were paid 100 £ at the beginning of the trial and 50 £ more if they completed all the trial. Besides, we estimate the financial risk taken by consumers adopting dynamic ToU tariff. We computed for each consumer of the control group the electricity bill with the two possible tariffs, the standard flat tariff and the dynamic ToU tariff. We found that the consumers were facing a risk with a statistically significant standard-deviation of 23 £ at the 5% level. Using the relation between the risk-premium (150 £) and the risk level (23 £) in the relation giving the certainty equivalent of a risk of known standard deviation for an exponential utility function, we estimated an absolute risk-aversion $r = 0.56$ per pound, which is very close to the producer's risk aversion parameter.
Consumption variation cost h. This parameter is related to the flexibility of the producer's generation capacities. The higher the flexibility of the generation, the lower the variance of consumption induces costs. With the development of intermittent energy sources, the quantification of the flexibility of a given power system has raised the attention of academics. For a review of this topic, we refer to Hirth (2015) [19]. The value for h depends on the whole electric system, and not only on the capacity of a single power plant. There is a difference between the flexibility of the all hydraulic electric system of Norway compared to the mix of German electric system based on wind generation and coal–fired plants. Nevertheless, if we focus on peak period of the day were flexibility is provided by gas–fired plants, we can make use of the estimations that exist for the cost of flexibility provided by power plants (see Kumar et al. (2012) [28] and Oxera (2003) [35], Table 3.2 p. 8 and Van den Bergh and Delarue (2015) [41], Table IV). Estimates find consistent values of order of magnitude of 25 to 42 €/MW^2.h for gas fired plants, which is in general the technology used in peaking period of the day. Thus, we choose a nominal value of $h = 40$ €/MW^2.h to consider a not so flexible system in which there may be room for flexibility exchange.
Costs of effort on average consumption μ Because we do not have access to data at the usage level, we first consider a single average usage. In order to fix an estimate of μ, we interpret the LCL experiment as the implementation of our demand–side model when there is no control of responsiveness or volatility (see Proposition 7.1). The results in Table 1 gives an order of magnitude for the realized average consumption deviation reduction, given by the difference between the consumption deviation of the control group (13.9 W) and the consumption deviation from the dToU group (3.5). Thus, one can estimate an average reduction of 10.4 W. According to Proposition 7.1, the absolute value of the average consumption deviation is given by:

$$\frac{1}{T}\Big|\mathbb{E}\Big[\int_0^T X_t dt\Big]\Big| = \frac{1}{3}\Lambda\bar{\mu}|\delta|T^2.$$

Recalling that $T = 5.5$, we obtain the value $\bar{\mu} = 2.93\,10^{-5}$. With this value of the parameter μ, the reduction of 10.4 W is obtained at the expense of a reduction cost given by $\int_0^T \widehat{c}_1(z_t)dt = \frac{1}{6}\bar{\mu}\Lambda^2\delta^2T^3 =$ 1.0 pence.
Costs of effort on consumption volatility λ and η. We set $\eta = 1$ for simplicity. The Figure 3 shows the variations of the total cost of effort for average consumption reduction and volatility reduction as a function of the parameter value λ. The cost of volatility reduction is always lower than the cost for average consumption reduction. Among all possible λ to be picked, there is one that maximises the cost of volatility reduction, which corresponds to the though experiment described in the introduction of this

section. We find that $\lambda = 3.5\ 10^{-2}$. With this parameter value, we find a responsiveness cost of effort of 0.5 pence.

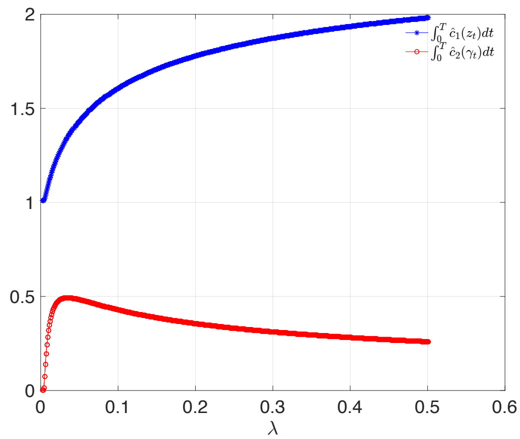

Figure 3: Cost of mean and volatility consumption deviation reduction as a function of λ.

T (h)	h (p/kW^{2}h)	δ (p/KWh)	p (p^{-1})	r (p^{-1})	σ (kW/h$^{1/2}$)	μ (kW^{2}h^{-1}p^{-1})	η	λ (p^{-1}kW^{2}h)
5.5	$4.0\ 10^{-4}$	-55.44	$0.6\ 10^{-2}$	$0.57\ 10^{-2}$	0.087	$2.93\ 10^{-5}$	1	$3.5\ 10^{-2}$

Table 2: Nominal values for model parameters.

Remark 7.1. (i) The consumption reduction cost of 1.0 p induces a reduction of 10 W for 5.5 hours. Thus, we find that there is a significant cost of action of order of 18 p/KWh, a value that ranges between the normal day price of 11.42 p/KWh and the high price during events of 67.2 p/KWh.
(ii) According to Proposition 3.3, the consumer makes an effort to reduce the volatility of his consumption when $(r\lambda\kappa^2(T-t)^2)^{-\frac{1}{2}} \leq 1$ when $\eta = 1$. With the values of the parameter calibrated above, one has $(r\lambda\kappa^2 T^2)^{-\frac{1}{2}} = 1.1$. Thus, the consumer does not exert effort without contract and the observed volatility before price events is σ.

7.3 Benefits from responsiveness incentive

We now examine the potential benefits of the implementation of a responsiveness incentive mechanism such as proposed by our model. We consider the case of a consumer with a single usage and linear energy value in the three possible situations: first–best (Proposition 4.1 and 6.3), second–best (Proposition 5.1 and 6.1) and second–best with uncontrolled responsiveness (Proposition 7.1). We summarize and make explicit in each case the certainty equivalent of the optimal value of the producer.

- *First best contracting*

$$\bar{v}(0,0) = \frac{1}{2}\int_0^T \Big(\mu\delta^2(T-t)^2 - \frac{\sigma^2}{\lambda}(\hat{b}(\gamma_t)^{-1} - 1) + \gamma_t \hat{b}(\gamma_t)\sigma^2\Big)dt$$

where $\gamma_t := -h - \rho\delta^2(T-t)^2$.

- *Second best contracting*

$$v(0,0) = \frac{1}{2}\int_0^T \Big(\mu\delta^2(T-t)^2 - \frac{\sigma^2}{\lambda}(\hat{b}(-q(z_{\mathrm{SB}}(t))^{-1}-1) - q(z_{\mathrm{SB}}(t)\hat{b}(-q(z_{\mathrm{SB}}(t))\sigma^2 - \mu(z_{\mathrm{SB}} - \delta(T-t))^2\Big)dt$$

where $q(z) := h + rz^2 + p(z-\delta(T-t))^2$ and $z_{\mathrm{SB}}(t)$ is the solution of

$$\inf_{z\in[\delta(T-t),\frac{p}{r+p}\delta(T-t)]} \frac{\sigma^2}{\lambda}(\hat{b}(-q(z))^{-1}-1) + q(z)\hat{b}(-q(z))\sigma^2 + \mu(z-\delta(T-t))^2.$$

- *Second best contracting with uncontrolled responsiveness*

$$w(0,0) = \frac{1}{2}\int_0^T \Big(\mu\delta^2(T-t)^2 - q(z_t)\sigma^2 - \mu(z_t - \delta(T-t))^2\Big)dt$$

where $z_t = \frac{p\sigma^2+\mu}{(p+r)\sigma^2+\mu}\,\delta(T-t)$.

Note that we have $w(0,0) \le v(0,0) \le \bar{v}(0,0)$.

	First-best	Second-best controled responsiveness	Second-Best uncontroled responsiveness
Reservation utility	-0.17	-0.17	-0.17
Payment for drift effort	1.22	1.34	1.01
Payment for volatility effort	0.20	0.49	0
Volatility risk premium	–	0.83	0.83
Total payment	3.36	1.89	0.57
Cost without contract	-5.69	-5.69	-5.69
Cost with contract	1.36	1.03	0.33
Gain from the contract	7.05	6.71	6.02
Volatility variation	-0.17	-0.18	0

Table 3: Volatility reduction, payments and benefits from the contract in pence. Payments in certainty equivalent, volatility variation in percentage.

Table 3 provides the consequences in terms of payments to the consumers and costs for the producer of the calibration of our model with the nominal parameters value summarized in Table 2. The terms used corresponds to the following quantities in certainty equivalent: Reservation utility L_0, Payment for the drift $\int_0^T \hat{c}_1(z_t)dt$, volatility risk premium $\int_0^T \frac{1}{2}z_t^2|\hat{\sigma}(\gamma_t)|^2dt$, Payment for volatility $\int_0^T \frac{1}{2}\hat{c}_2(\gamma_t)dt$, Payment for the energy value $\int_0^T f(X_t)dt$, Total payment Y_T, cost without contract $J_P(0,\mathbb{P}^\nu)$, cost with contract V^{SB}.

Because in the second–best with responsiveness control, the payment rate $z_{\mathrm{SB}}(t)$ on the drift also depends on λ when the consumer is incentivised to reduce volatility, the cost of average energy reduction raises from 1.0 when there is no responsiveness control to 1.3 pence when there is one. It is a significative increase, but does not alter the comparatives of the contracts with and without responsiveness control. Besides, the payment for the effort on volatility is one third of the payment for the reduction on the drift.

Thus, the consumer devotes a smaller part of his budget for responsiveness control, which gives ground to the idea that our calibration hypothesis for the parameter λ is not too much optimistic. We note that the volatility risk–premium is not a negligible part of the payment made to the consumer. Indeed, for a reduction effort that last five hours, this part of the payment is even larger than the payment for the efforts on volatility reduction. But, the risk–premium for volatility is the same with and without responsiveness control.

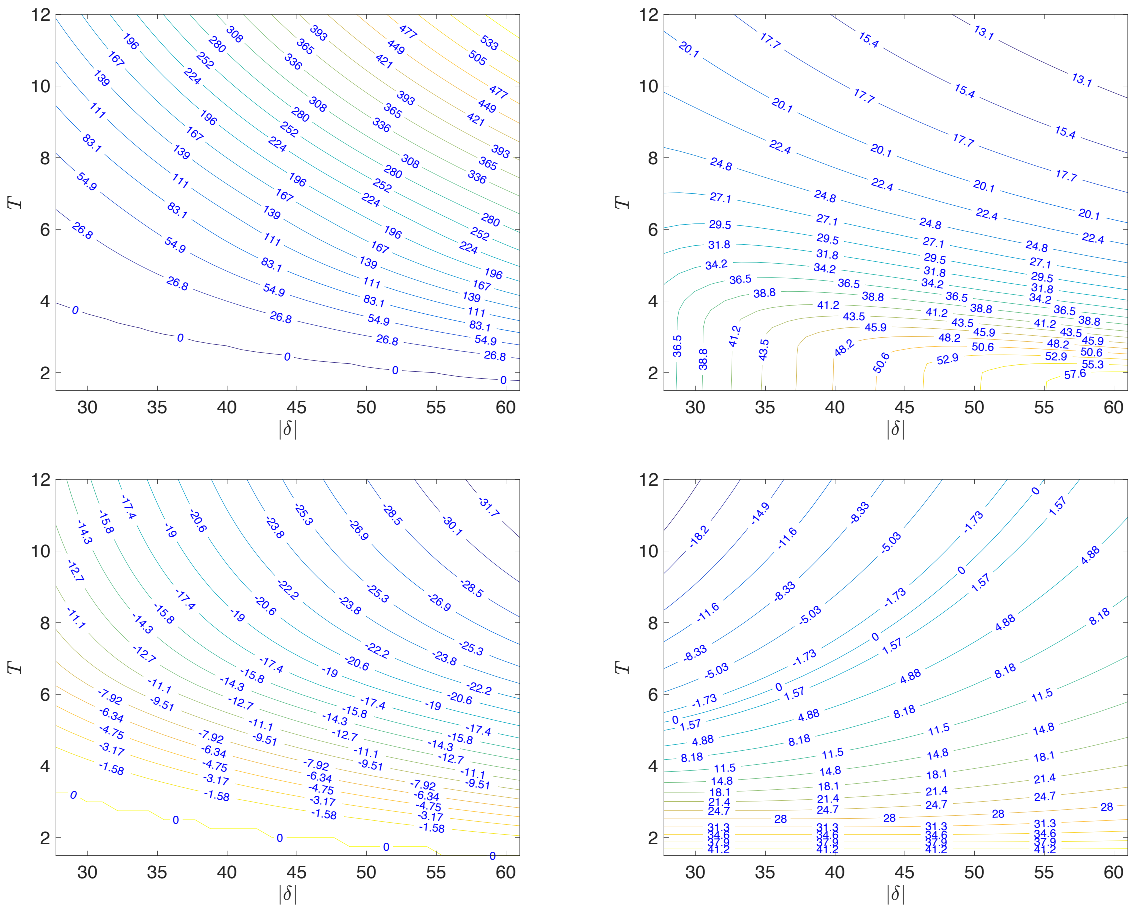

Figure 4: (Left to right) Gain from responsiveness control, volatility variation between second-best and no-contract, loss between first-best and second-best as a function of the price event duration T and the absolute value of the energy value discrepancy δ. All in percentage.

We now turn to the gain from the responsiveness control. The producer is able to achieve a positive certainty equivalent three times larger when implementing responsiveness control. Indeed, the gain from responsiveness control as well as the information rent is sensitive to both the duration of the price event T and the energy value discrepancy δ, since the certainty equivalent of the value function of the producers grows proportionally to $\delta^2 T^3$. We observe also a significant decrease of volatility of approximately 18%.

The Figure 4 presents the sensitivity analysis of the gain from responsiveness control, the loss of wellfare as measured by the information rent and the variation of the volatility as functions of T and δ. We considered shorter and longer price event up to 12 hours and considered situations with lower energy value discrepancy. We observe that the lower the energy value discrepancy, the longer the price event should be to ensure a significant benefit of responsiveness control. For instance, for short term price event below four

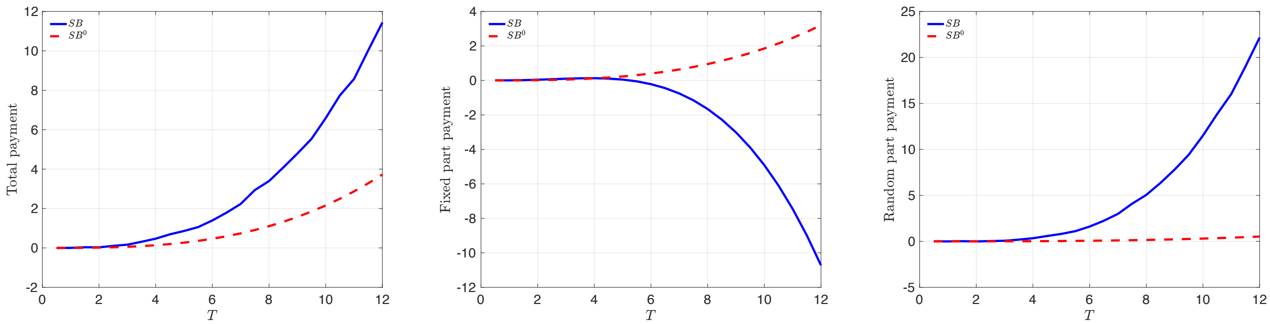

Figure 5: Total (left), fixed part (middle) and certainty equivalent of the random part (right) of the optimal payment with responsiveness control (blue) and without (red) as a function of the price event duration T in pence.

hours, the energy value discrepancy should be at least 35 p/kWh to observe a gain from responsiveness control (Figure 4, up left). The incentive on volatility needs time or a large energy value discrepancy to show its benefits. This point is also made clear when looking at the variation of volatility between the case of the second-best and the case without contract (Figure 4, down left). One observes the same pattern for the volatility reduction as the economical gain from responsiveness control. Further, the region of the space (T, δ) that produces the highest benefits from responsiveness control are also the situations where there is the less well-fare loss even (Figure 4, up left). This proximity of the first and second best may be attributed to the fact that there is only one usage. Further, if the consumer has the potentiality to make efforts on the volatility and this potentiality is not used, one can observe an increase of the volatility (Figure 4, down right).

Figure 5 shows the total payment and its decomposition between its fixed part and its part depending on the efforts of the consumer as a function of the duration of the price event T with and without responsiveness control. In both cases, the total payment is positive and increases when the duration of the effort becomes large. It is not clear on the graphic because of scaling effect but the total payment is non-monotonic: it begins by decreasing for low event duration, reaches a minimum and then starts to monotonically increase. As no surprise, the payment with responsiveness control is larger than without it because it requires more efforts from the consumers. The remarkable result comes from the decomposition of the contract between its deterministic part and its random part that depends on the realised consumption reduction. For small duration, the deterministic part is positive with an without responsiveness control. But, for long duration, the fixed part of the payment with responsiveness control becomes negative while the random part is 10 times larger than the random part without responsiveness control. When the duration of the effort is long (for instance, 12 hours), the producer first takes 10 pence out of the consumer and then, gives him back 25 pence if he has made the expected effort. If the reader authorizes a comparison with the basic incentive mechanism of the carrot and the stick, the optimal contract without responsiveness control only relies on a increasing carrot for an increasing duration of effort (the fixed part is positive and increases with large T). But, the optimal contract with responsiveness control relies on both the carrot and the stick for a long duration of effort. First, the Principal takes money from the Agent (the stick) and then, she gives back a much larger amount of money in case of successful efforts (the carrot). We cannot resist the temptation of making this result a general principle: if one wants to induce regular results from an agent on a long term basis, one should first reduce his income compared to his peers but then, pay him much more in case of success.

7.4 Robustness of the linear contract

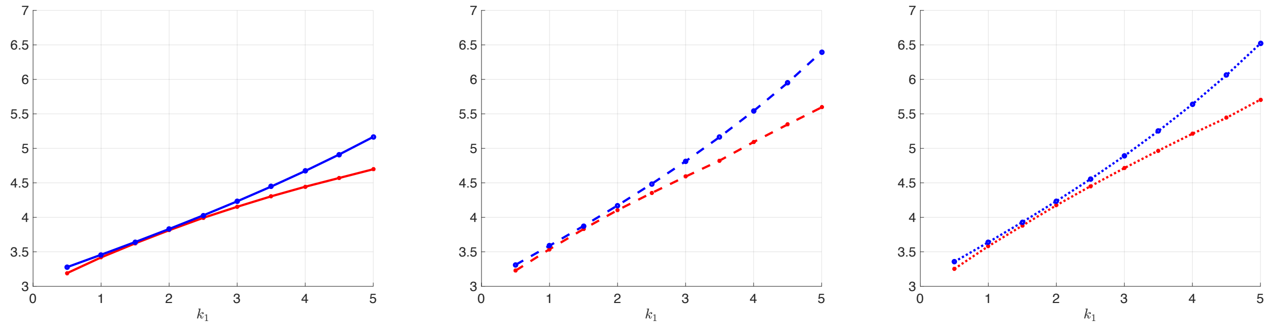

Figure 6: Expected payment with one (Left), two (Middle) and four (Right) usages with the linear approximation of f (red stars) and with the non-linear f (blue dots).

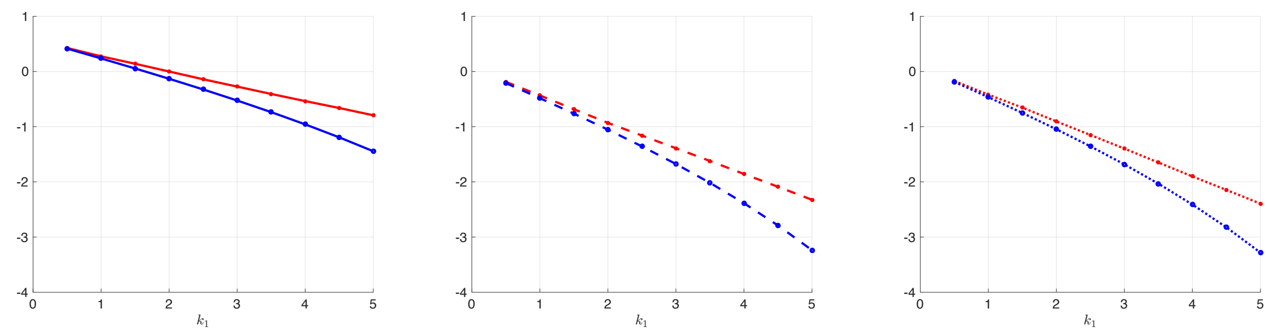

Figure 7: Producer certainty equivalent value with one (Left), two (Middle) and four (Right) usages with the linear approximation of f (red stars) and with the non-linear f (blue dots).

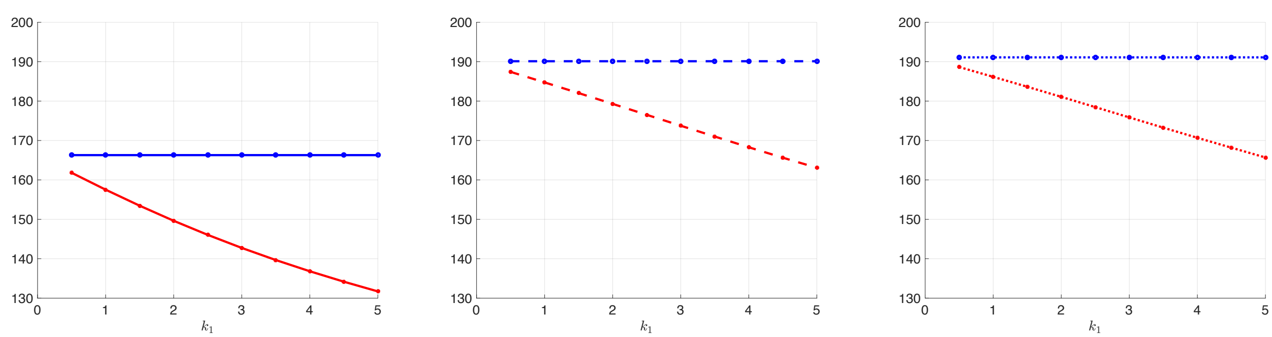

Figure 8: Volatilities (Watt) with one (Left), two (Middle) and four (Right) usages with the linear approximation of f (blue dots) and with the non-linear f (red stars).

We check in this section the robustness of the hypothesis of a linear value of energy. For sake of simplicity, we concentrate on the effect of a decreasing marginal value of energy for the consumer and

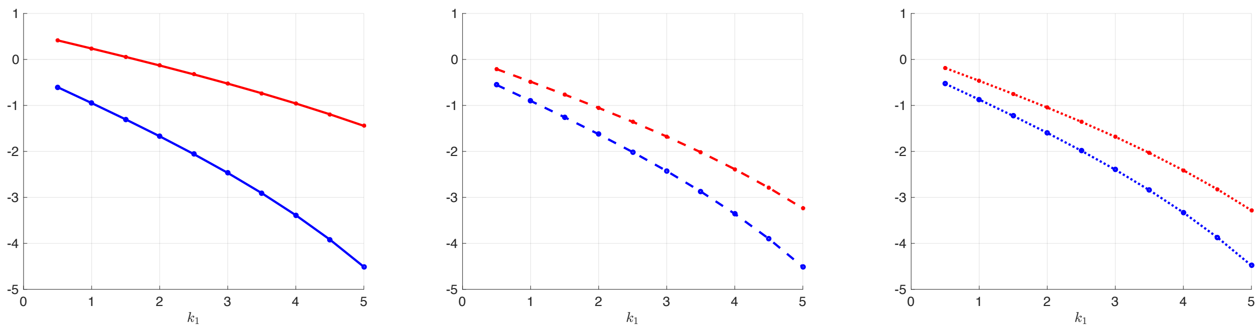

Figure 9: Producer certainty equivalent value from responsiveness control with the linear approximation contract with one (Left), two (Middle) and four (Right) usages with the linear approximation of f (blue dots) and with the non-linear f (red stars).

leave aside its counterpart on the generation side (increasing marginal cost of generation). We consider now the following specification of the function f:

$$f(x) = \kappa \frac{1 - e^{-k_1 x}}{k_1}, \tag{7.2}$$

so that, for small values of κ_1, we recover the linear case with $f(x) \approx \kappa x$.

Further, we calibrated our model for a single average usage. But, in the case of a single usage, the producer can identify the effort on the responsiveness and thus, be close to the first-best. This is no longer the case when there are more usages. Thus, we assess also the mean payment and the benefit of the contract for the producer in the context of two and four usages. In this case we split the parameters μ, λ and σ of the nominal situation provided by Table 2 with the vector of weight $(1/4 \;\; 3/4)$ for the two-usage case and $(1/8 \;\; 1/8 \;\; 1/2 \;\; 1/4)$ for the four-usage case. The choice of the vector of weight is guided by the idea of making a contrasted difference between usages.

Figures 6, 7 and 8 provides respectively the expected payment to the consumer, the certainty equivalent gain from the contract for the producer and the volatilities as a function of the concavity of f measured by k_1. Red curves corresponds to the numerical solution of the PDE of the second-best case with responsiveness control in Proposition 5.1 (i). The PDE was solved using a standard finite difference method together with an implicit–Euler scheme. Blue curves correspond to the implementation of the contract defined by the payment rates given by Proposition 6.1 (i). In both cases, the initial condition of the contract is given by the reservation utility of the consumer given by Proposition 3.2 with f being given by relation (7.2).

Figure 6 shows that expected payment with the linear approximation of f remains close to the exact payment if the concavity is not too strong. Beyond a value of k_1 of 3, the producer overpays the customer because she overestimates the value of energy for the consumer. This overestimation leads to a loss in the contract gains (Figure 7) and a lower volatility reduction than what can be achieved with the correct consumer's value of energy function (Figure 8). A five fold multiplication of the concavity of f leads to a reduction by 20% of the contract benefit for the producer.

Figure 9 presents the certainty equivalent gain of the producer in the case of the implementation of the linear contract with responsiveness and without responsiveness as a function of the concavity of the energy value of the consumer. The responsiveness control mechanism still provides a significant benefit to the producer. The pure effect of concavity leads to a worst case decrease of the gain to 70% while the joint effects of concavity and multiple usages leads to a worst case decrease of the gain to 30%, for 2 or 4 usages.

8 Conclusion

We presented in this paper a new point of view on the demand response contract using the moral hazard problem in Principal–Agent framework, and showed how it makes it possible to reduce the average consumption, while improving the responsiveness of the consumer. We provided a closed–form expression for the optimal contract in the case of linear energy value discrepancy and showed that the optimal contract allows the system to bear more risk. The calibration of our model to pricing trial data predicts that the cost of efforts of the consumer to reduce his average consumption will lead to non negligible benefits for producers. Using a conservative hypothesis, we predict that consumer's responses to price events would be significantly higher and that the electric system would significantly benefit from the implementation of a responsiveness incentive mechanism. These predictions are testable in standard pricing trial experiment like the Low Carbon London experiment. If our claim is true, a simple premium indexed to the regularity of consumption accross price events should deeply enhance the efficiency of demand response programs.

A Technical proofs

A.1 Proof of Proposition 3.2

(i) Since f is increasing, the consumer has no reason to make an effort on the drift of consumption deviation, as no compensation is offered for this costly effort. However, this argument does not apply to the effort on the volatility, due to the consumer's risk aversion. As the consumer has constant risk aversion utility with parameter r, her reservation utility reduces to

$$R_0 := \sup_{\mathbb{P}^{(0,\beta)}\in\mathcal{P}} \mathbb{E}^{\mathbb{P}^{(0,\beta)}}\Big[-e^{-r\int_0^T (f(X_t)-\frac{1}{2}c_2(\beta_t))dt}\Big]. \tag{A.1}$$

The concavity of R_0 in the initial data X_0 is a direct consequence of the concavity of f and the convexity of c_2. By standard stochastic control theory, it follows that $R_0 = R(0, X_0)$, where the function R is the dynamic version of the reservation utility, with final value $R(T, x) = -1$, and is a viscosity solution of the corresponding HJB equation:

$$0 = \partial_t R - rfR + \sup_{b\in(0,1]^d} \frac{1}{2}\big(|\sigma(b)|^2 R_{xx} + rc_2(b)R\big) = \partial_t R - rfR - rR\,H_{\rm v}\big(\frac{R_{xx}}{-rR}\big).$$

Denote by $X_s^{t,x} := x + X_{s-t}$ the shifted canonical process started from initial data (t, x). As f is Lipschitz, notice that

$$\begin{aligned} R(t,x) \ge \mathbb{E}^{\mathbb{P}^{0,1}}\Big[-e^{-r\int_t^T f(X_s^{t,x})ds}\Big] &\ge -\mathbb{E}^{\mathbb{P}^{0,1}}\Big[e^{r|f'|_\infty \int_t^T |X_s^{t,x}|ds}\Big] \\ &\ge -\mathbb{E}^{\mathbb{P}^{0,1}}\Big[e^{r|f'|_\infty((T-t)|x|+\int_t^T |X_s^{t,0}|dt)}\Big] \ge -C_1 e^{r|f'|_\infty(T-t)|x|}, \end{aligned}$$

where

$$C_1 := \mathbb{E}^{\mathbb{P}^{0,1}}\Big[e^{r|f'|_\infty \int_0^T |X_s^{t,0}|ds)}\Big] < \infty,$$

since $X_s^{t,0}$ is a centred Gaussian random variable for all $s \in [t,T]$. As $c_2 \ge 0$, we also have

$$R(t,x) \le \sup_{\mathbb{P}^{(0,\beta)}\in\mathcal{P}} \mathbb{E}^{\mathbb{P}^{0,\beta}}\Big[-e^{-r\int_t^T f(X_s^{t,x})ds}\Big] = \mathbb{E}^{\mathbb{P}^{0,1}}\Big[-e^{-r\int_t^T f(X_s^{t,x})ds}\Big] \le -C_2 e^{-r|f'|_\infty(T-t)|x|},$$

where

$$C_2 := \mathbb{E}^{\mathbb{P}^{0,1}}\Big[e^{-r|f'|_\infty \int_0^T |X_s^{t,0}|ds)}\Big] < \infty,$$

by the same argument as previously.

Then, the certainty equivalent function E, defined by $R =: -e^{-rE}$, satisfies the PDE (3.11), and has growth controlled by $E(t,x) \le (C_1 \vee C_2)(T-t)|x|$.

(ii) We now assume that the PDE (3.11) has a $C^{1,2}$ solution E with growth controlled by $|E(t,x)| \le C(T-t)|x|$. Then $\hat{R} := -e^{-rE}$ is also $C^{1,2}([0,T]\times\mathbb{R})$. Denote $K_t^\beta := e^{-r\int_0^t (f(X_s)-\frac{1}{2}c_2(\beta_s))ds}$, and $T_n := \inf\{t>0 : |X_t - X_0| \ge n\}$, we compute by Itô's formula that for all $\mathbb{P}^{(0,\beta)} \in \mathcal{P}$,

$$\begin{aligned}\hat{R}(0,X_0) &= \mathbb{E}^{\mathbb{P}^{0,\beta}}\Big[K_{T_n}^\beta \hat{R}(T_n, X_{T_n}) - \int_0^{T_n} K_t^\beta\Big(\partial_t\hat{R} + \frac{1}{2}|\sigma(\beta_t)|^2 v_{xx} - r\Big(f - \frac{1}{2}c_2(\beta_t)\Big)\hat{R}\Big)(t,X_t)dt\Big]\\ &\ge \mathbb{E}^{\mathbb{P}^{0,\beta}}\Big[K_{T_n}^\beta \hat{R}(T_n, X_{T_n})\Big] \longrightarrow \mathbb{E}^{\mathbb{P}^{0,\beta}}\Big[K_T^\beta \hat{R}(T,X_T)\Big] = \mathbb{E}^{\mathbb{P}^{0,\beta}}\Big[-K_T^\beta\Big],\end{aligned}$$

where the local martingale part verifies that $\mathbb{E}^{\mathbb{P}^{0,\beta}}\big[\int_0^{T_n} K_t^\beta \hat{R}_x(t,X_t)\sigma(\beta_t)dW_t\big] = 0$, by the fact that $\hat{R}_x$ is bounded on $[0,T_n]$ and $\sigma(\beta)$ is bounded. The second inequality follows from the PDE satisfied by $\hat{R}$, and the last limit is obtained by using the control on the growth of R together with the final condition $\hat{R}(T,.) = -1$. By the arbitrariness of $\mathbb{P}^{(0,\beta)} \in \mathcal{P}$, this implies that $\hat{R}(0,X_0) \ge R_0$.

To prove equality, we now observe from Proposition 3.1 that by choosing $b^0(t,x) := \widehat{b}(E_{xx} - E_x^2)(t,x)$ as defined in Proposition 3.1, the (unique) inequality in the previous calculation is turned into an equality provided that the stochastic differential equation $dX_t = \sigma(b(t,X_t))dW_t$ has a weak solution. This, in turn, is implied by the fact that the function $\sigma \circ b$ is bounded and continuous, see Karatzas and Shreve [27, Theorem 5.4.22 and Remark 5.4.23]. Consequently, $\hat{R}(0,X_0) = R_0$, and (a^0, b^0) are optimal feedback controls. □

A.2 Proof of Proposition 4.1

By standard stochastic control theory, $\bar{V}$ can be characterized by means of the corresponding HJB equation

$$\begin{cases}\partial_t\bar{V} - \rho(f-g)\bar{V} + \sup\limits_{a\in\mathbb{R}_+^d}\big\{-a\cdot\mathbf{1}\bar{V}_x + \rho c_1(a)\bar{V}\big\} + \dfrac{1}{2}\sup\limits_{b\in[0,1)^d}\big\{|\sigma(b)|^2(\bar{V}_{xx} + \rho h\bar{V}) + \rho c_2(b)\big\} = 0\\ \bar{V}(T,.) = -1.\end{cases}$$

Setting $\bar{V}(t,x) = -e^{-\rho\bar{v}(t,x)}$, we obtain by direct substitution the PDE satisfied by $\bar{v}$

$$\begin{cases}-\partial_t\bar{v} = (f-g) - \inf\limits_a\big\{a\cdot\mathbf{1}\bar{v}_x + c_1(a)\big\} - \dfrac{1}{2}\inf\limits_b\big\{c_2(b) - |\sigma(b)|^2\big(\bar{v}_{xx} - \rho\bar{v}_x^2 - h\big)\big\}\\ \bar{v}(T,x) = 0,\end{cases}\tag{A.2}$$

which coincides with the PDE in the proposition statement, by definition of the consumer's Hamiltonian (3.9). We next prove the control on the growth of v. First, as the cost function c is non-negative, we have

$$\bar{V} \le \sup_{\mathbb{P}^\nu}\mathbb{E}^{\mathbb{P}^\nu}\Big[-e^{-\rho(\int_0^T (f-g)(X_t)dt)}\Big] \le \sup_{\mathbb{P}^\nu}\mathbb{E}^{\mathbb{P}^\nu}\Big[-e^{-\rho((f-g)(X_0)+|f'-g'|_\infty\int_0^T |X_t|dt)}\Big] < -\infty.$$

On the other hand, as the cost of no-effort $c(0,1) = 0$, it follows that

$$\bar{V} \ge \mathbb{E}^{\mathbb{P}^{0,1}}\Big[-e^{-\rho(\int_0^T (f-g)(X_t)dt)}\Big] \ge \sup_{\mathbb{P}^\nu}\mathbb{E}^{\mathbb{P}^\nu}\Big[-e^{-\rho((f-g)(X_0)-|f'-g'|_\infty\int_0^T |X_t|dt)}\Big] > -\infty.$$

This shows that $e^{C_1 T|X_0|} \leq \bar{V} \leq e^{C_2 T|X_0|}$, for some constants $C_1, C_2 > 0$, and therefore $|E(0, X_0)| \leq (C_1 \vee C_2)|x|T$. By homogeneity of the problem, we deduce the announced control on growth by simply removing the time origin to any $t < T$.

Under smoothness assumptions, we follow the line of the verification argument of the proof of Proposition 3.2 to prove that the optimal consumer's response derived in Proposition 3.1 is an optimal feedback control for the problem $\bar{V}$. Using the fact that $V^{\mathrm{FB}} = R\big(\frac{1}{R} e^{-\rho(\bar{v}(0,X_0))}\big)^{\frac{p+r}{r}}$ and noting $L_0 := -\frac{1}{r}\log(-R)$, one gets

$$V^{\mathrm{FB}} = -e^{-p(\bar{v}(0,X_0)-L_0)}.$$

Finally, the expression of ξ_{FB} follows by direct substitution of the optimal Lagrange multiplier (4.3) in (4.2). □

A.3 Proof of Lemma 5.1

Recall that $f_0(q,\gamma) = q\big|\widehat{\sigma}(\gamma)\big|^2 + \widehat{c}_2(\gamma)$ where

$$\widehat{c}_2(\gamma) = \sum_{j=1}^{d} \frac{\sigma_j^2}{\eta_j \lambda_j}\Big(\widehat{b}_j(\gamma)^{-\eta_j} - 1\Big),\ \big|\widehat{\sigma}(\gamma)\big|^2 = \sum_{j=1}^{d} \sigma_j^2 \widehat{b}_j(\gamma) \text{ and } \widehat{b}_j(\gamma) := 1 \wedge (\lambda_j \gamma^-)^{-\frac{1}{1+\eta_j}} \vee \varepsilon.$$

If $\lambda_j\gamma^- \leq 1$ then $\widehat{b}_j(\gamma) = 1$ and thus $\widehat{b}_j'(\gamma) = 0$. Similarly, if $\lambda_j\gamma^- \geq \varepsilon^{-(1+\eta_j)}$, then $\widehat{b}_j(\gamma) = \varepsilon$ and thus $\widehat{b}_j'(\gamma) = 0$. Finally, if $\varepsilon^{-(1+\eta_j)} > \lambda_j\gamma^- > 1$ then $\widehat{b}_j(\gamma) = (-\lambda_i\gamma)^{-\frac{1}{1+\eta_i}}$ and thus

$$\widehat{b}_j'(\gamma) = \frac{1}{1+\eta_j}\lambda_j^{-\frac{1}{1+\eta_j}}(-\gamma)^{-\frac{1}{1+\eta_j}-1} = -\frac{1}{\gamma(1+\eta_j)}\widehat{b}_j(\gamma).$$

Define now

$$f_j(\gamma) := \frac{\sigma_j^2}{\eta_j\lambda_j}\big(\widehat{b}_j(\gamma)^{-\eta_j} - 1\big) + q\sigma_j^2\widehat{b}_j(\gamma).$$

We have

$$f_j'(\gamma) = \frac{\sigma_j^2}{\eta_j\lambda_j}\Big(-\eta_j\widehat{b}_j(\gamma)^{-\eta_j-1}\widehat{b}_j'(\gamma)\Big) + q\sigma_j^2\widehat{b}_j'(\gamma).$$

If $-\varepsilon^{-(1+\eta_j)} < \lambda_j\gamma < -1$, one has $\widehat{b}_j(\gamma)^{-\eta_j-1} = -\lambda_j\gamma$ and thus

$$f_j'(\gamma) = \sigma_j^2(\gamma+q)\widehat{b}_j'(\gamma) = -\Big(1+\frac{q}{\gamma}\Big)\frac{\sigma_j^2}{1+\eta_j}\widehat{b}_j(\gamma)\mathbf{1}_{\{-\varepsilon^{-(1+\eta_j)}<\lambda_j\gamma<-1\}}.$$

Hence

$$\frac{\partial f_0}{\partial\gamma}(q,\gamma) = -\Big(1+\frac{q}{\gamma}\Big)\sum_{i=1}^{d}\frac{\sigma_i^2}{1+\eta_i}\widehat{b}_i(\gamma)\mathbf{1}_{\{-\varepsilon^{-(1+\eta_j)}<\lambda_i\gamma<-1\}}.$$

Therefore, the minimum of $f_0(q,\gamma)$ over negative γ is reached either at $\gamma = -q$ when there is at least one index $i \in \{1,\ldots,d\}$ such that $q \in [1/\lambda_i, \varepsilon^{-(1+\eta_i)}/\lambda_i]$, and otherwise f_0 does not depend on q. Hence, we can consider that the minimiser is always $\gamma = -q$, and thus $F_0(q) = f_0(q,-q)$.

Direct computations now show that

$$f_j(-q) = \frac{\sigma_j^2}{\lambda_j}\Bigg(\mathbf{1}_{\{\lambda_j q\leq 1\}}\lambda_j q + \mathbf{1}_{\{\varepsilon^{-(1+\eta_j)}>\lambda_j q>1\}}\frac{(1+\eta_j)(\lambda_j q)^{\frac{\eta_j}{1+\eta_j}} - 1}{\eta_j} + \mathbf{1}_{\{\varepsilon^{-(1+\eta_j)}\leq\lambda_j q\}}\Big(\lambda_j\varepsilon q + \frac{\varepsilon^{-\eta_j}-1}{\eta_j}\Big)\Bigg),$$

so that by adding all the terms

$$F_0(q) = \sum_{j=1}^{d} \frac{\sigma_j^2}{\lambda_j}\bigg(\mathbf{1}_{\{\lambda_j q \le 1\}}\lambda_j q + \mathbf{1}_{\{\varepsilon^{-(1+\eta_j)} > \lambda_j q > 1\}} \frac{(1+\eta_j)(\lambda_j q)^{\frac{\eta_j}{1+\eta_j}} - 1}{\eta_j} + \mathbf{1}_{\{\varepsilon^{-(1+\eta_j)} \le \lambda_j q\}}\bigg(\lambda_j \varepsilon q + \frac{\varepsilon^{-\eta_j} - 1}{\eta_j}\bigg)\bigg),$$

from which it is clear that F_0 is non–decreasing. $\square$

A.4 Proof of Proposition 5.1

By standard stochastic control theory, the dynamic version of the value function of the Principal, denoted by $V(t,x,\ell) := V^{\rm SB}(t,x,\ell)$, is a viscosity solution, with appropriate growth at infinity, of the corresponding HJB equation

$$-\partial_t V = (g-f)V_\ell + \frac{1}{2}\sup_{(z,\gamma)\in\mathbb{R}^2}\Big\{\big|\widehat{\sigma}(\gamma)\big|^2\big[(h+rz^2)V_\ell + V_{xx} + z^2 V_{\ell\ell} + 2zV_{x\ell}\big] - \bar{\mu}\big(2z^- V_x - (z^-)^2 V_\ell\big) + \widehat{c}_2(\gamma)V_\ell\Big\},$$

with terminal condition $V(T,x,\ell) = U(-\ell)$, for $(x,\ell)\in\mathbb{R}^2$. Under the constant relative risk aversion specification of the utility function of the producer, it follows that

$$-p\partial_t v = p(f-g) - \frac{1}{2}\inf_{z,\gamma}\Big\{\big|\widehat{\sigma}(\gamma)\big|^2\big((h+rz^2)p - pv_{xx} + p^2(v_x)^2 + z^2p^2 - 2zp^2 v_x\big)$$
$$+ \bar{\mu}\big(2z^- p v_x + p(z^-)^2\big) + p\widehat{c}_2(\gamma)\Big\},$$

which reduces to the PDE (5.3).

The control on the growth of the function v is deduced from the control on the growth of V by following the same line of argument as in the proof of Proposition 4.1, using the Lipschitz feature of the difference $f-g$. Similarly, under smoothness condition, the same verification argument leads to the optimal feedback controls, defined as the maximisers of the second-best producer's Hamiltonian, which determine the optimal payment rates.

We finally verify that the additional properties of the optimal payment rates hold. First, if $v_x \ge 0$, the map $z \longmapsto F_0\big(h - v_{xx} + rz^2 + p(z-v_x)^2\big) + \bar{\mu}(z^- + v_x)^2\big\}$ is non–increasing for $z \le \frac{p}{r+p}v_x$ because F_0 is a non–decreasing function. Thus, the minimum of the map is reached for the minimum of $q(v_x, v_{xx}, z)$ which is $z_{\rm SB} = \frac{p}{r+p}v_x$. Second, if $v_x \le 0$, the preceding map is non-longer monotonic on the interval $(v_x, \frac{p}{r+p}v_x)$. But, it is non–increasing for $z \le v_x$ and non–decreasing for $z \ge \frac{p}{r+p}v_x$, making its infimum lie between v_x and $\frac{p}{r+p}v_x$. In both cases, the optimiser with respect to γ can be deduced from Lemma 5.1, and is given by (5.4). $\square$

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