On continuous time contract theory

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Outline

(Static) Principal-Agent Problem

- Principal delegates management of output process X,
  only observes X

- Agent devotes effort $a \Longrightarrow X^a$, chooses optimal effort by

$$V_A := \max_a \mathbb{E}\, U_A\big(\qquad - c(a)\big)$$

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  pays salary defined by contract $\xi(X)$

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$$V_A(\xi) := \max_a \mathbb{E}\, U_A\big(\xi(X^a) - c(a)\big) \quad \Longrightarrow \quad \hat{a}(\xi)$$

- Principal chooses optimal contract by solving

$$\max_\xi \mathbb{E}\, U_P\big(X^{\hat{a}(\xi)} - \xi(X^{\hat{a}(\xi)})\big) \quad \text{under constraint} \quad V_A(\xi) \geq \rho$$

$\Longrightarrow$ Non-zero sum Stackelberg game

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Principal-Agent problem formulation

Agent problem :

$$V_0^A(\xi) \;:=\; \sup_{\mathbb{P}\in\mathcal{P}} \mathbb{E}^{\mathbb{P}}\Big[\xi(X) - \int_0^T c_t(\nu_t)dt\Big]$$

$\mathbb{P}\in\mathcal{P}$: weak solution of Output process for some ν valued in U :

$$dX_t \;=\; b_t(X,\nu_t)dt + \sigma_t(X,\nu_t)dW_t^{\mathbb{P}} \quad \mathbb{P}-\text{a.s.}$$

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$$V_0^P \;:=\; \sup_{\xi\in\Xi_\rho} \mathbb{E}^{\mathbb{P}^*(\xi)}\Big[U\big(\ell(X)-\xi(X)\big)\Big]$$

$$\text{where} \quad \Xi_\rho \;:=\; \big\{\xi(X) : \; V_0^A(\xi)\geq\rho\big\}$$

Possible extensions : random (possibly ∞) horizon, heterogeneous agents with possibly mean field interaction, competing Principals...

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Principal-Agent problem formulation : non-degeneracy

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GENERAL SOLUTION APPROACH

A subset of revealing contracts

- Path-dependent Hamiltonian for the Agent problem :

$$H_t(\omega, z, \gamma) \quad := \quad \sup_{u \in U} \left\{ b_t(\omega, u) \cdot z + \frac{1}{2} \sigma_t \sigma_t^\top (\omega, u) : \gamma - c_t(\omega, u) \right\}$$

- For $Y_0 \in \mathbb{R}$, Z, Γ $\mathbb{F}^X$ – prog meas, define $\mathbb{P}$–a.s. for all $\mathbb{P} \in \mathcal{P}$

$$Y_t^{Z,\Gamma} = Y_0 + \int_0^t Z_s \cdot dX_s + \frac{1}{2} \Gamma_s : d\langle X \rangle_s - H_s(X, Z_s, \Gamma_s) ds$$

Proposition $V_A(Y_T^{Z,\Gamma}) = Y_0$. Moreover $\mathbb{P}^*$ is optimal iff

$$\nu_t^* = \operatorname*{Argmax}_{u \in U} H_t(Z_t, \Gamma_t) = \hat{\nu}(Z_t, \Gamma_t)$$

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Principal problem restricted to revealing contracts

Dynamics of the pair (X, Y) under "optimal response"

$$dX_t = \underbrace{\nabla_z H_t(X, Y_t^{Z,\Gamma}, Z_t, \Gamma_t)}_{b_t(X,\hat{\nu}(Y_t,Z_t,\Gamma_t))} dt + \underbrace{\left\{2\nabla_\gamma H_t(X, Y_t^{Z,\Gamma}, Z_t, \Gamma_t)\right\}^{\frac{1}{2}}}_{\sigma_t(X,\hat{\nu}(Y_t,Z_t,\Gamma_t))} dW_t$$

$$dY_t^{Z,\Gamma} = Z_t \cdot dX_t + \frac{1}{2}\Gamma_t : d\langle X\rangle_t - H_t(X, Y_t^{Z,\Gamma}, Z_t, \Gamma_t)dt$$

$\Longrightarrow$ Principal's value function under revealing contracts :

$$V_P \geq V_0(X_0, Y_0) := \sup_{(Z,\Gamma)\in\mathcal{V}} \mathbb{E}\Big[U\big(\ell(X) - Y_T^{Z,\Gamma}\big)\Big], \quad \text{for all } Y_0 \geq \rho$$

where $\mathcal{V} := \Big\{(Z,\Gamma):\ Z \in \mathbb{H}^2(\mathcal{P}) \text{ and } \mathcal{P}^*\big(Y_T^{Z,\Gamma}\big) \neq \emptyset\Big\}$

Reduction to standard control problem

Theorem (Cvitanić, Possamaï & NT '15)

Assume $\mathcal{V} \neq \emptyset$. Then

$$V_0^P = \sup_{Y_0 \geq \rho} V_0(X_0, Y_0)$$

Given maximizer $Y_0^\star$, the corresponding optimal controls $(Z^\star, \Gamma^\star)$ induce an optimal contract

$$\xi^\star = Y_0^\star + \int_0^T Z_t^\star \cdot dX_t + \frac{1}{2}\Gamma_t^\star : d\langle X\rangle_t - H_t(X, Y_t^{Z^\star,\Gamma^\star}, Z_t^\star, \Gamma_t^\star)dt$$

Recall the subclass of contracts

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$$\mathbb{P} - \text{a.s. for all } \mathbb{P} \in \mathcal{P}$$

To prove the main result, it suffices to prove the **representation**

$$\text{for all } \xi \in ?? \quad \exists\, (Y_0, Z, \Gamma) \quad \text{s.t.} \quad \xi = Y_T^{Z,\Gamma}, \ \mathbb{P} - \text{a.s. for all } \mathbb{P} \in \mathcal{P}$$

OR, weaker sufficient condition :

$$\text{for all } \xi \in ?? \quad \exists\, (Y_0^n, Z^n, \Gamma^n) \quad \text{s.t.} \quad \text{“} Y_T^{Z^n,\Gamma^n} \longrightarrow \xi \text{”}$$

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Connexion with nonlinear parabolic PDEs

Consider the Markov case $\xi = g(X_T)$:

$$Y_T = g(X_T), \text{ and } dY_t = Z_t \cdot dX_t + \frac{1}{2}\Gamma_t : d\langle X\rangle_t - H_t(X_t, Y_t, Z_t, \Gamma_t)dt$$

Intuitively, $Y_t = v(t, X_t)$ with decomposition (Itô's formula)

$$dY_t = \partial_t v(t, X_t)dt + Dv(t, X_t) \cdot dX_t + \frac{1}{2}D^2 v(t, X_t) : d\langle X\rangle_t$$

By direct identification : $Z_t = Dv(t, X_t)$, $\Gamma_t = D^2 v(t, X_t)$, and

$$\partial_t v + H(., v, Dv, D^2 v) = 0, \text{ with boundary cond. } v\big|_{t=T} = g$$

Representation $\equiv$ path-dependent nonlinear parabolic PDE

Outline

1. The Principal-Agent problem
   - Formulation
   - Reduction to standard control problem
2. Fully nonlinear representation in random horizon
   - Semimartingale measures on the canonical space
   - Random horizon 2nd order backward SDEs

Canonical space

$\Omega := \{\omega \in C^0(\mathbb{R}_+, \mathbb{R}^d) : \omega(0) = 0\}$

X : canonical process, i.e. $X_t(\omega) := \omega(t)$

$\mathcal{F}_t := \sigma(X_s, s \le t)$, $\mathbb{F} := \{\mathcal{F}_t, t \ge 0\}$

$\mathcal{P}^W$: collection of all semimartingale measures $\mathbb{P}$ such that

$$dX_t = b_t dt + \sigma_t dW_t, \quad \mathbb{P} - \text{a.s.}$$

for some $\mathbb{F}-$processes b and σ, and $\mathbb{P}-$Brownian motion W

Class of prob. meas. on Ω : $\mathcal{P} \subset \mathcal{P}^W$, sufficiently rich...

$\mathcal{P}-$quasi-surely MEANS $\mathbb{P}-$a.s. for all $\mathbb{P} \in \mathcal{P}$

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Quadratic variation process

$\langle X \rangle$: quadratic variation process (defined on $\mathbb{R}_+ \times \Omega$)

$$\langle X \rangle_t := X_t^2 - \int_0^t 2X_s dX_s = \mathbb{P}-\lim_{|\pi|\to 0} \sum_{n\geq 1} \left| X_{t\wedge t_n^\pi} - X_{t\wedge t_{n-1}^\pi} \right|^2$$

for all $\mathbb{P} \in \mathcal{P}^W$, and set

$$\hat{\sigma}_t^2 := \overline{\lim}_{h\searrow 0} \frac{\langle X \rangle_{t+h} - \langle X \rangle_t}{h}$$

Example : $(d = 1)$ Let $\mathbb{P}_1$:=Wiener measure, i.e. X is a $\mathbb{P}_1-$BM, and define $\mathbb{P}_2 := \mathbb{P}_1 \circ (2X)^{-1}$. Then

$$\hat{\sigma}_t = 1, \quad \mathbb{P}_1 - \text{a.s. and} \quad \hat{\sigma}_t = 2, \quad \mathbb{P}_2 - \text{a.s.}$$

$\mathbb{P}_1$ and $\mathbb{P}_2$ are singular

Nonlinear expectation operators

$\mathcal{P}^0$: subset of local martingale measures, i.e.

$$dX_t = \sigma_t dW_t, \quad \mathbb{P}-\text{a.s. for all} \quad \mathbb{P} \in \mathcal{P}^0$$

and

$$\mathcal{P}^L := \cup_{\mathbb{P}\in\mathcal{P}^0} \mathcal{P}^L(\mathbb{P}) \quad \text{where} \quad \mathcal{P}^L(\mathbb{P}) := \big\{\mathbb{Q} = \mathbf{D}(\lambda)\cdot\mathbb{P} : \ \|\lambda\|_{\mathbb{L}^\infty} \le L\big\}$$

$\mathbf{D}(\lambda)$: the Doléans-Dade exponential $\frac{d\mathbf{D}(\lambda)_t}{\mathbf{D}(\lambda)_t} = \lambda_t \cdot dW_t$, $\mathbf{D}(\lambda)_0 = 1$

$\Longrightarrow$ Nonlinear expectations

$$\mathcal{E}_t^{\mathbb{P}} := \sup_{\mathbb{Q}\in\mathcal{P}^L(\mathbb{P})} \mathbb{E}_t^{\mathbb{Q}}, \quad \mathcal{E}_t := \sup_{\mathbb{P}\in\mathcal{P}^0} \mathbb{E}_t^{\mathbb{P}}, \quad \text{and} \quad \mathcal{E}_t^L := \sup_{\mathbb{P}\in\mathcal{P}^L} \mathbb{E}_t^{\mathbb{P}}$$

$\mathbb{Q} \sim \mathbb{P}$ on $\mathcal{F}_T$ for all $T \in \mathbb{R}+$ and $\mathbb{Q} \not\sim \mathbb{P}$ on $\mathcal{F}_\infty$ in general

Random horizon 2^{nd} order backward SDE

For a stop. time τ, and $\mathcal{F}_\tau-$measurable ξ :

$$Y_{t\wedge\tau} = \xi + \int_{t\wedge\tau}^{\tau} F_s(Y_s, Z_s, \hat{\sigma}_s)ds - \int_{t\wedge\tau}^{\tau} Z_s \cdot dX_s + \int_{t\wedge\tau}^{\tau} dK_s, \quad \mathcal{P} - \text{q.s.}$$

K non-decreasing, $K_0 = 0$, and minimal in the sense

$$\inf_{\mathbb{P}\in\mathcal{P}} \mathbb{E}^{\mathbb{P}}\Big[\int_{t_1\wedge\tau}^{t_2\wedge\tau} dK_t\Big] = 0, \quad \text{for all} \quad t_1 \le t_2$$

- "(Y, Z) supersolution of BSDE($\mathbb{P}$) for all $\mathbb{P} \in \mathcal{P}$ "
- "(Y, Z) solution of BSDE($\mathbb{P}^*$) for some $\mathbb{P}^* \in \mathcal{P}$ "

Connection with fully nonlinear PDEs

Rewrite the 2BSDE in differential form

$$dY_t = -F_t(Y_t, Z_t, \hat{\sigma}_t)dt + Z_t \cdot dX_t - dK_t, \;\; t \le \tau, \quad \text{and}\, Y_T = \xi, \;\; \mathcal{P} - \text{q.s.}$$

Markovian case $\xi = g(X_T)$ and $F_t(X, y, z, \hat{\sigma}_t) = f(t, X_t, y, z, \hat{\sigma}_t) \Longrightarrow$ $Y_t = v(t, X_t)$

$$dY_t = \partial_t v(t, X_t)dt + Dv(t, X_t) \cdot dX_t + \frac{1}{2}\mathrm{Tr}\big[\hat{\sigma}_t^2 D^2 v(t, X_t)\big]\, dt, \;\; \mathcal{P} - \text{q.s.}$$

by Itô's formula. Direct identification yields

$$Z_t = Dv(t, X_t) \quad \text{and} \quad \partial_t v(t, X_t) + \frac{1}{2}\mathrm{Tr}\big[\hat{\sigma}_t^2 D^2 v(t, X_t)\big] \le -F_t(v, Dv, \hat{\sigma}_t)$$

Finally, the minimality condition on K implies the fully nonlinear PDE

$$\partial_t v(t, X_t) + \sup_{\sigma}\Big\{\frac{1}{2}\mathrm{Tr}\big[\sigma^2 D^2 v(t, X_t)\big] + F_t(v, Dv, \sigma)\Big\} = 0$$

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Nonlinearity

Assumptions $F : \mathbb{R}_+ \times \omega \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}_+^d \longrightarrow \mathbb{R}$ satisfies

(C1$_L$) Lipschitz in $(y, \sigma z)$:

$$\big|F(.,y,z,\sigma) - F(.,y',z',\sigma)\big| \le L \left(|y-y'| + \big|\sigma(z-z')\big|\right.$$

(C2$_\mu$) Monotone in y :

$$(y-y') \cdot \big[F(.,y,.) - F(.,y',.)\big] \le -\mu\, |y-y'|^2$$

Denote $f_t(y,z) := F_t\big(y,z,\widehat{\sigma}_t\big)$ and $f_t^0 := F_t\big(0,0,\widehat{\sigma}_t\big)$

Remark Deterministic finite horizon $\tau = T$: $(\mathrm{C2})_\mu$ not needed
Soner, NT & Zhang '14 and Possamaï, Tan & Zhou '16

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Assumptions $F : \mathbb{R}_+ \times \omega \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}^d_+ \longrightarrow \mathbb{R}$ satisfies

(C1$_L$) Lipschitz in $(y, \sigma z)$:

$$\left|F(., y, z, \sigma) - F(., y', z', \sigma)\right| \leq L \left(|y - y'| + \left|\sigma(z - z')\right|\right.$$

(C2$_\mu$) Monotone in y :

$$(y - y') \cdot \left[F(., y, .) - F(., y', .)\right] \leq -\mu \, |y - y'|^2$$

Denote $f_t(y, z) := F_t\big(y, z, \widehat{\sigma}_t\big)$ and $f^0_t := F_t\big(0, 0, \widehat{\sigma}_t\big)$

Remark Deterministic finite horizon $\tau = T$: $(\mathrm{C2})_\mu$ not needed
Soner, NT & Zhang '14 and Possamaï, Tan & Zhou '16

Wellposedness of random horizon 2^{nd}order backward SDE

Theorem (Y. Lin , Z. Ren, NT & J. Yang '17)

Let $\|\xi\|_{\mathcal{L}^q_{\rho,\tau}(\mathcal{P}^L)} < \infty$, $\overline{f}^q_{\rho,\tau} := \mathcal{E}^L\Big[\Big(\int_0^\tau \big|e^{\rho t} f^0_t\big|^2 ds\Big)^{\frac{q}{2}}\Big]^{\frac{1}{q}} < \infty$, *for some* $\rho > -\mu$, $q > 1$. *Then the Random horizon 2BSDE has a unique solution* (Y, Z) *with*

$$Y \in \mathcal{D}^p_{\eta,\tau}(\mathcal{P}^L), \quad Z \in \mathcal{H}^p_{\eta,\tau}(\mathcal{P}^L) \quad \textit{for all} \quad \eta \in [\mu, \rho), \ p \in [1, q)$$

$$\|\xi\|^p_{\mathcal{L}^q_{\rho,\tau}(\mathcal{P})} := \mathcal{E}^L\Big[\big|e^{\rho\tau}\xi\big|^q\Big], \quad \|Y\|^p_{\mathcal{D}^p_{\eta,\tau}(\mathcal{P})} := \mathcal{E}^L\Big[\sup_{t\le\tau}\big|e^{\eta t} Y_t\big|^p\Big]$$

$$\|Z\|^p_{\mathcal{H}^p_{\eta,\tau}(\mathcal{P})} := \mathcal{E}^L\Big[\Big(\int_0^\tau \big|e^{\eta t}\widehat{\sigma}_t^{\mathrm{T}} Z_t\big|^2 dt\Big)^{\frac{p}{2}}\Big]$$

Back to Principal-Agent problem

Recall the subclass of contracts

$$Y_t^{Z,\Gamma} = Y_0 + \int_0^t Z_s \cdot dX_s + \frac{1}{2}\Gamma_s : d\langle X\rangle_s - H_s(X, Y_s^{Z,\Gamma}, Z_s, \Gamma_s)ds, \quad \mathcal{P} - \text{q.s.}$$

To prove the main result, it suffices to prove the **representation**

$$\text{for all } \xi \in ?? \quad \exists\, (Y_0, Z, \Gamma) \quad \text{s.t.} \quad \xi = Y_T^{Z,\Gamma}, \; \mathcal{P} - \text{q.s.}$$

OR, weaker sufficient condition :

$$\text{for all } \xi \in ?? \quad \exists\, (Y_0^n, Z^n, \Gamma^n) \quad \text{s.t.} \quad \text{“} Y_T^{Z^n,\Gamma^n} \longrightarrow \xi \text{”}$$

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Reduction to second order BSDE

- $H_t(\omega, y, z, \gamma)$ non-decreasing and convex in γ, Then

$$H_t(\omega, y, z, \gamma) = \sup_{\sigma \geq 0} \left\{ \frac{1}{2}\sigma^2 : \gamma - H_t^*(\omega, y, z, \sigma) \right\}$$

Denote $k_t := H_t(Y_t, Z_t, \Gamma_t) - \frac{1}{2}\hat{\sigma}_t^2 : \Gamma_t + H_t^*(Y_t, Z_t, \hat{\sigma}_t) \geq 0$

Then, required representation $\xi = Y_T^{Z,\Gamma}$, $\mathcal{P}-$q.s. is **equivalent to**

$$\xi = Y_0 + \int_0^T Z_t \cdot dX_t + H_t^*(Y_t, Z_t, \hat{\sigma}_t)dt - \int_0^T k_t dt, \quad \mathcal{P} - \text{q.s.}$$

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