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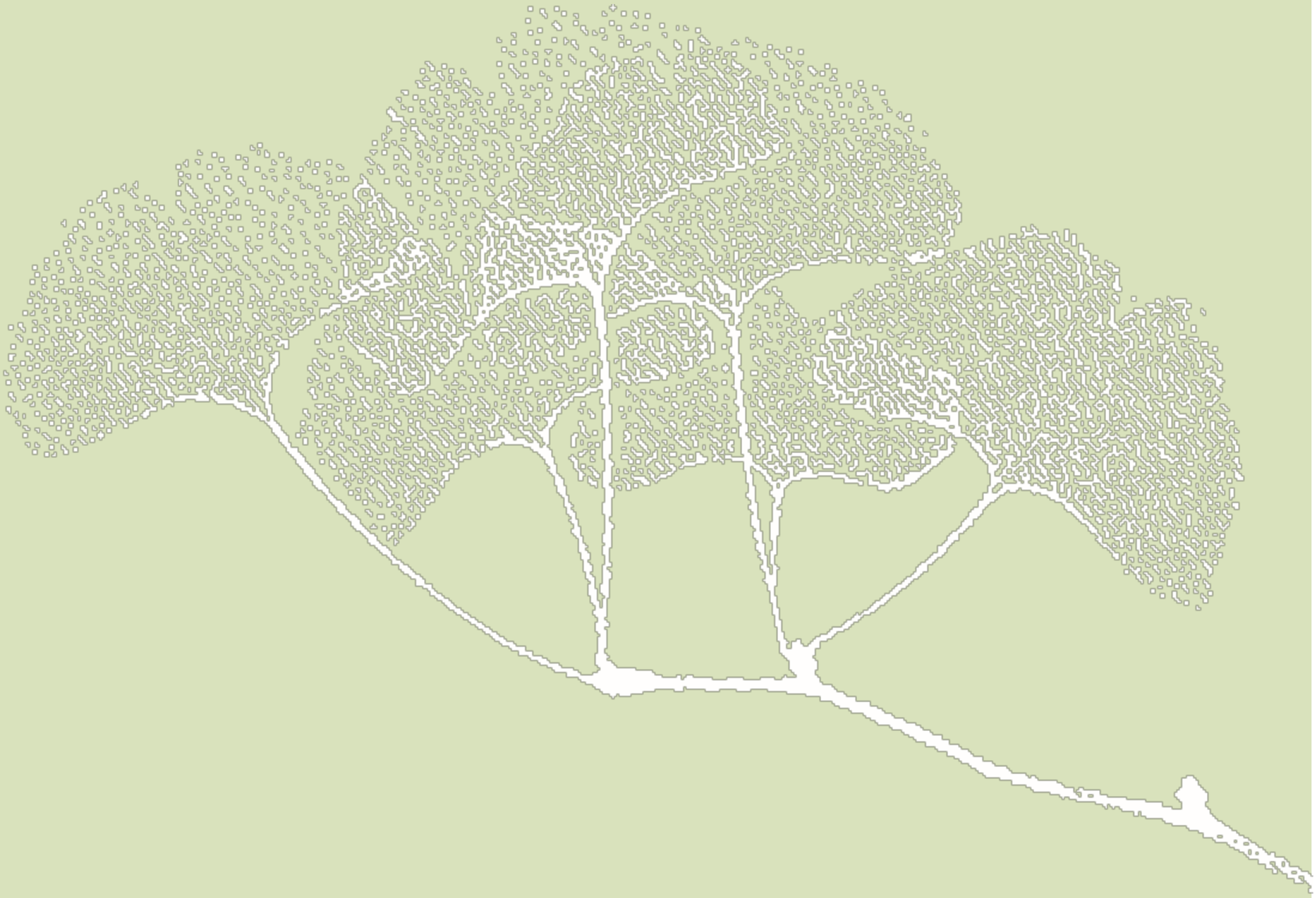

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Continuous-Time Heterogeneous Agent Models with Recursive Utility and Preference for Late Resolution

Yves Achdou[*], Qing Tang[†]

Abstract

We consider continuous-time heterogeneous agent models with recursive utility (Epstein-Zin utility) cast as mean field games, in which agents prefer late resolution of uncertainty. The model leads to a system coupling a pair of Hamilton-Jacobi-Bellman equations with state constraints and Fokker-Planck-Kolmogorov equations. We investigate the existence of solutions to the mean field game system and discuss some important qualitative features of the model.

1 Introduction

In this paper we extend the study of continuous-time heterogeneous agent models done in Achdou, Han, Lasry, Lions, and Moll [3] by addressing the Epstein-Zin recursive utility. The continuous-time formulation of the Aiyagari-Bewley-Huggett models [26, 4, 30], classical in recursive macroeconomics, and the related system of partial differential equations can be studied in the light of the mathematical theory of mean field games (cf. [1, 29]). Such models involve a large number of *ex ante* identical but *ex post* heterogeneous agents in an incomplete market setting. Each agent maximizes her/his utility over time with consumption decisions, facing debt limits and idiosyncratic income risks. She/he solves the stochastic optimal control problem, taking as given an equilibrium interest rate r:

$$v(x,y) = \max_{c_\tau} \mathbb{E}\left[\int_t^\infty f(c_\tau, v_\tau)\,d\tau \big| \boldsymbol{x}_t = x,\, \boldsymbol{y}_t = y\right], \text{ subject to } \left\{ \begin{array}{lcl} d\boldsymbol{x}_\tau &=& (r\boldsymbol{x}_\tau + \boldsymbol{y}_\tau - c_\tau)d\tau,\ \tau > t, \\ \boldsymbol{x}_\tau &\geq& \underline{x}. \end{array} \right. \tag{1.1}$$

Here, $\boldsymbol{x}_\tau$ and $\boldsymbol{y}_\tau$ respectively stand for the agent's wealth and labor income at time τ. The control variable is the consumption c_τ and the flow f depends also on the future value $v_\tau = v(\boldsymbol{x}_\tau, \boldsymbol{y}_\tau)$. It is assumed that $\boldsymbol{y}_\tau$ is a Poisson process with two states $y_1 < y_2$. The Epstein-Zin recursive utility is defined as:

$$f(c,v) = \frac{\rho}{1-\psi^{-1}} \frac{c^{1-\psi^{-1}} - ((1-\gamma)v)^\theta}{((1-\gamma)v)^{\theta-1}}, \qquad \theta = \frac{1-\psi^{-1}}{1-\gamma}, \tag{1.2}$$

where ρ is the subjective discount rate. It is assumed that ψ, the elasticity of intertemporal substitution (EIS) and γ, the risk aversion parameter, are both positive and do not take the value 1. This type of nonexpected, recursive utility was proposed in [18] in discrete time and continuous time models were introduced in [16] and [17]. In the present paper, we use the same notation

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as in more recent economic literature [32, 37]. A key feature of the Epstein-Zin utility (1.2) is the separation between risk aversion γ and elasticity of intertemporal substitution (EIS) ψ. The time-additive separable CRRA utility is a special case of recursive utility in which $\gamma = \psi^{-1}$ and

$$f(c, v) = \frac{\rho c^{1-\gamma}}{1-\gamma} - \rho v. \tag{1.3}$$

It was observed in [18, p. 952] that the attitude towards the timing of the resolution of uncertainty is pinned down by the constant $\gamma\psi$: early (late) resolution is preferred if $\gamma\psi > (<)1$. With $\gamma\psi = 1$, the agent is indifferent to the timing of uncertainty resolution. In this paper, we only consider the case $\frac{1-\psi^{-1}}{1-\gamma} > 1$ with $\gamma > 1$, and the results can be extended to the case $\frac{1-\psi^{-1}}{1-\gamma} > 1$ with $\gamma < 1$. The extension to the case $\gamma\psi = 1$ is easy. Our assumption thus implies $\gamma\psi < 1$. In this context, it is more straightforward to use the viscosity solution theory and have a complete theory. Therefore, the theoretical results in this paper are only applicable to the case when agents prefer late resolution.

We use the shorthand notation $v_t = v(\boldsymbol{x}_t, \boldsymbol{y}_t)$ when needed. Since $\boldsymbol{y}_t$ takes two values it is convenient to set $v_j(x) = v(x, y_j)$. The agent's optimization problem (1.1) leads to the weakly coupled system of Hamilton-Jacobi-Bellman (HJB) equations for the value functions $v_j(x)$:

$$0 = \max_{c \geq 0} \{f(c, v_j) + (rx + y_j - c)Dv_j\} + \lambda_j(v_{\bar{j}}(x) - v_j(x)), \quad j \in \{1, 2\} \quad \text{and} \quad \bar{j} = 3 - j, \tag{1.4}$$

with a state constraint boundary condition at $\underline{x}$. In the spirit of "Bewley models", the interest rate r is determined in equilibrium and depends on the aggregate wealth and labor in the economy. To close the model, we need to consider the distribution of agents by wealth and income. We introduce a stationary Fokker-Planck-Kolmogorov (FPK) equation, which describes the invariant measure m_j of agents with income y_j. The measure m_j has a density g_j and possibly exhibits a Dirac mass at $\underline{x}$ weighted by μ_j, and the density satisfies:

$$-\frac{\partial}{\partial x}\left[(rx + y_j - c_j(x))g_j(x)\right] + \lambda_{\bar{j}} g_{\bar{j}}(x) - \lambda_j g_j(x) = 0. \tag{1.5}$$

The aggregate capital supply and labor are denoted by

$$\begin{aligned} K[m] &= \int_{x \geq \underline{x}} x dm_1 + \int_{x \geq \underline{x}} x dm_2 = \int_{x \geq \underline{x}} x g_1(x) dx + \int_{x \geq \underline{x}} x g_2(x) dx + \sum_{j \in \{1,2\}} \mu_j \underline{x}, \\ N[m] &= \int_{x \geq \underline{x}} y_1 dm_1 + \int_{x \geq \underline{x}} y_2 dm_2 = \int_{x \geq \underline{x}} y_1 g_1(x) dx + \int_{x \geq \underline{x}} y_2 g_2(x) dx + \sum_{j \in \{1,2\}} \mu_j y_j. \end{aligned} \tag{1.6}$$

In the Huggett model [26], each agent can borrow or lend at interest rate r while the aggregate capital is fixed to the value B. The recursive equilibrium in a stationary Huggett model is then summarized by the system

$$\begin{cases} (i) & 0 = \max_{c \geq 0} \{f(c, v_j) + (r^* x + y_j - c)Dv_j(x)\} + \lambda_j(v_{\bar{j}}(x) - v_j(x)), \\ & c_j^*(x) = \arg\max_{c \geq 0} \{f(c, v_j) + (r^* x + y_j - c)Dv_j(x)\}, \\ (ii) & -\dfrac{\partial}{\partial x}\left[(rx + y_j - c_j^*(x))g_j(x)\right] + \lambda_{\bar{j}} g_{\bar{j}}(x) - \lambda_j g_j(x) = 0, \\ & \displaystyle\int_{x \geq \underline{x}} g_1(x) dx + \int_{x \geq \underline{x}} g_2(x) dx + \sum_{j \in \{1,2\}} \mu_j = 1, \end{cases} \tag{1.7}$$

with the equilibrium r determined by the implicit coupling condition

$$(iii_H) \qquad \mathcal{K}[r^*] = K[m^{(r^*)}] = B. \tag{1.8}$$

In the Aiyagari model (1.11), the asset x is interpreted as homogeneous physical capital. The total factor of productivity (TFP) and the capital depreciation rate are respectively denoted by A and δ. The production in the economy is described using the Cobb-Douglas production function $\mathbf{F}$ such that $\mathbf{F}(K, N) = AK^\alpha N^{1-\alpha}$ with $0 < \alpha < 1$ and where K is the aggregate capital and N is the aggregate labor. The profit of the producer is $\mathbf{F}(K, N) - (r + \delta)K - wN$ where w is the wage. Here we normalize the wage to unity, i.e. $w = 1$. Then, at any given interest rate r, the producer chooses a level of capital demand $\mathcal{K}_d[r]$ such that

$$\mathcal{K}_d[r] = \arg\max_K \left\{ \mathbf{F}(K, N) - (r + \delta)K \right\}. \tag{1.9}$$

It can be deduced from state constraints that under any given interest rate r,

$$\mu_j + \int_{x>\underline{x}} g_j(x)dx = \frac{\lambda_{\bar{j}}}{\lambda_j + \lambda_{\bar{j}}}, \quad N[m] = \frac{y_{\bar{j}}\lambda_j}{\lambda_j + \lambda_{\bar{j}}} + \frac{y_j\lambda_{\bar{j}}}{\lambda_j + \lambda_{\bar{j}}}. \tag{1.10}$$

The stationary Aiyagari model is described by an equilibrium interest rate r^* such that $\mathcal{K}[r^*] = \mathcal{K}_d[r^*]$, i.e. supply equals demand for capital. This leads one to supplement system (1.7) with the coupling condition:

$$(iii_A) \qquad r^* = A\alpha \left(\frac{\mathcal{K}[r^*]}{N} \right)^{\alpha-1} - \delta = A\alpha \left(\frac{K[m^{(r^*)}]}{N} \right)^{\alpha-1} - \delta. \tag{1.11}$$

Continuous-time consumption-saving models with recursive utility have been used for studying financial consequence of disasters [32], dynamic portfolio choice [15] and economics of climate change [25]. In particular, such a model with state constraints (incomplete market setting) and stochastic income process has been considered in [37].

Discrete-time recursive utility models have been widely used for macroeconomic asset pricing [19, 22, 23] and were considered in a more general mathematical framework of abstract dynamic programming [34, 33]. There is a vast literature on the existence and uniqueness of the value function for recursive utility process in discrete time, e.g. [8] and [9]. The connection between discrete-time and continuous-time models with recursive utilities has been addressed in [27].

It was proposed in [3] that the suitable notion of solution to the HJB equation in these heterogeneous agent systems is the *constrained viscosity solution* [12]. Since then, several works [10, 24, 35] have contributed to the mathematical analysis of such models with time-additive utilities. In this paper, we extend the *constrained viscosity solution* theory to HJB equations with recursive utility and this allows us to propose monotone numerical schemes (monotonicity in the sense of Barles-Souganidis [6]). Numerical methods based on finite difference schemes have been proposed in [3] for the same kind of models with CRRA utilities. Recently, a semi-Lagrangian scheme was proposed in [10]. Both methods are well adapted for the models addressed in the present paper.

This paper is organized as follows. In Section 2 we give some preliminaries and recall useful notions in convex analysis. Section 3 is devoted to the analysis of HJB equations. We discuss the existence and uniqueness of the *constrained viscosity solution*, and prove that the latter is the value function of the recursive optimal control problem. Higher regularity of the solution is then obtained using in particular arguments from convex analysis. We establish some key features of the optimal saving policies. In Section 4 we consider the solution to the FPK equation given $r < \rho$. In Section 5 we first study the HJB equation and saving policies with $r = \rho$ and prove the nonexistence of an invariant measure in this case. This leads to the blow up of the aggregate wealth as $r \to \rho$. We then address the existence of an equilibrium for the Aiyagari model. Finally, we give some numerical examples in Section 6.

2 Preliminaries

The assumptions that follow will be made in the whole paper:

$$\gamma > 1,\ 0 < \psi < 1,\ \gamma\psi < 1. \tag{2.1}$$

$$\rho \geq r, \quad \rho \underline{x} + y_j > 0.$$

An equivalent statement to (2.1) would be $\gamma > 1$ and $\theta > 1$.

With Epstein-Zin utility (1.2), we can rewrite the HJB equation (1.4)

$$\begin{aligned} \frac{\rho}{\theta} v_j(x) &= \max_{c \geq 0} \{\mathcal{F}(c, v_j) + (rx + y_j - c)Dv_j\} + \lambda_j (v_{\bar{j}}(x) - v_j(x)) \\ &= H(x, y_j, v_j, Dv_j) + \lambda_j (v_{\bar{j}}(x) - v_j(x)), \end{aligned} \tag{2.2}$$

where we use the notation

$$f(c, v_j) = \mathcal{F}(c, v_j) - \frac{\rho}{\theta} v_j, \qquad \mathcal{F}(c, v_j) = \frac{\rho}{1 - \psi^{-1}} \frac{c^{1-\psi^{-1}}}{((1-\gamma)v_j)^{\theta-1}}. \tag{2.3}$$

The Hamiltonian in (2.2) is

$$H(x, y, v, p) = \begin{cases} (rx + y)p + \dfrac{\rho^{\psi}}{\psi - 1} p^{1-\psi} ((1-\gamma)v)^{\frac{1-\gamma\psi}{1-\gamma}}, & \text{if} \quad p \geq 0, \\ +\infty, & \text{if} \quad p < 0. \end{cases} \tag{2.4}$$

We summarize the notation in Table 1.

It is clear with (2.1) that for any $c > 0$, $\mathcal{F}(c, v)$ is decreasing in v since

$$\mathcal{F}_v(c, v) < 0. \tag{2.5}$$

For all $c > 0$ and $v < 0$, the second order derivatives of the aggregator $f(c, v)$ and the determinants of Hessian matrices satisfy

$$\begin{aligned} &f_{cc}(c, v) = \mathcal{F}_{cc}(c, v) = \frac{-\rho c^{-\psi^{-1}-1}}{\psi((1-\gamma)v)^{\theta-1}} < 0, \quad f_{vv}(c, v) = \mathcal{F}_{vv}(c, v) = (\gamma - \psi^{-1}) \frac{\rho c^{1-\psi^{-1}}}{((1-\gamma)v)^{\theta+1}} < 0, \\ &\det \begin{pmatrix} f_{cc}(c, v) & f_{cv}(c, v) \\ f_{vc}(c, v) & f_{vv}(c, v) \end{pmatrix} = \rho^2 \gamma (\psi^{-1} - \gamma) \frac{c^{-2\psi^{-1}}}{((1-\gamma)v)^{2\theta}} > 0. \end{aligned} \tag{2.6}$$

Therefore $f(c, v)$ and $\mathcal{F}(c, v)$ are jointly concave in (c, v).

Table 1: Symbols

Subjective discount factor	$\rho > 0$
Risk aversion	$\gamma > 1$
Elasticity of intertemporal substitution (EIS)	$0 < \psi < 1$
A parameter arising in Epstein-Zin utility	$\theta := \frac{1-\psi^{-1}}{1-\gamma} > 1$
A convenient parameter	$b := \rho \left[\frac{r+\psi(\rho-r)}{\rho}\right]^{\frac{1}{1-\psi}}$
Scaled discount	$\zeta = \rho/\theta$
Aggregator	$f(c, v)$
Modified aggregator	$\mathcal{F}(c, v) = f(c, v) + \zeta v$
Borrowing limit	$\underline{x}$
Equilibrium interest rate	$r^* < \rho$

The optimal consumption (away from the borrowing limit) is

$$c_j = \arg\max_{c \geq 0} \{\mathcal{F}(c, v_j) - cDv_j\} = \rho^{\psi}(Dv_j)^{-\psi}((1-\gamma)v_j)^{\frac{1-\gamma\psi}{1-\gamma}}. \tag{2.7}$$

We observe that $H(x, y, v, p)$ defined in (2.4) is convex in p for fixed (x, y, v) with $p > 0$,

$$\min_{p>0} H(x, y, v, p) = \rho \frac{(rx+y)^{1-\psi^{-1}}}{1-\psi^{-1}} \big((1-\gamma)v\big)^{\frac{\psi^{-1}-\gamma}{1-\gamma}}. \tag{2.8}$$

Moreover, since $rx + y > 0$ and $p^{1-\psi}$ is sublinear, we infer from (2.4) the coercivity condition

$$H(x, y, v, p) \to +\infty \quad \text{when} \quad p \to +\infty. \tag{2.9}$$

We denote by $H_v(x, y, v, p)$ and $H_{vv}(x, y, v, p)$ the first and second order derivatives of $H(x, y, v, p)$ with respect to v. We use $H_{vp}(x, y, v, p)$ to denote the cross second order derivative of $H(x, y, v, p)$ with respect to v and p. We obtain from (2.1):

- $H(x, y, v, p)$ is decreasing in the $v-$ variable:

$$H_v(x, y, v, p) = \frac{\rho^{\psi}(1-\gamma\psi)}{\psi - 1} p^{1-\psi}((1-\gamma)v)^{\frac{\gamma(1-\psi)}{1-\gamma}} < 0, \tag{2.10}$$

- $H_v(x, y, v, p)$ is decreasing in both the $v-$ and $p-$ variables:

$$H_{vp}(x, y, v, p) = -\rho^{\psi}(1-\gamma\psi)p^{-\psi}((1-\gamma)v)^{\frac{\gamma(1-\psi)}{1-\gamma}} < 0. \tag{2.11}$$

$$H_{vv}(x, y, v, p) = -\gamma\rho^{\psi}(1-\gamma\psi)p^{1-\psi}((1-\gamma)v)^{\frac{\gamma(1-\psi)}{1-\gamma}-1} < 0. \tag{2.12}$$

From (2.12), we deduce that $H(x, y, v, p)$ is strictly concave in the v variable.

We will also use the following elementary but important inequality whose proof is left to the reader.

Lemma 2.1. *For $\rho > r$, the parameter b introduced in Table 1 is such that $r < b < \rho$.*

Let us recall some notions from convex analysis.

Definition 2.2. Consider $\phi \in C[\underline{x}, +\infty)$. The super- and the subdifferential of ϕ at x are given by

$$\begin{aligned} &(1) \quad D^+\phi(x) := \left\{ p \in \mathbb{R} : \limsup_{z \to x,\, z \geq \underline{x}} \frac{\phi(z) - \phi(x) - p(z-x)}{|z-x|} \leq 0 \right\}, \\ &(2) \quad D^-\phi(x) := \left\{ p \in \mathbb{R} : \liminf_{z \to x,\, z \geq \underline{x}} \frac{\phi(z) - \phi(x) - p(z-x)}{|z-x|} \geq 0 \right\}. \end{aligned} \tag{2.13}$$

$D^+\phi(x)$ and $D^-\phi(x)$ are closed and convex subsets of $\mathbb{R}$. Moreover, it is clear that if ϕ is nondecreasing in a neighborhood of x then $D^+\phi(x)$ and $D^-\phi(x)$ are subsets of $\mathbb{R}_+$.

For a locally Lipschitz function ϕ, we define the set (cf. [5, p. 63])

$$D^*\phi(x) = \left\{ p \in \mathbb{R} : p = \lim_{n \to +\infty} D\phi(x_n),\ x_n \to x \right\}, \tag{2.14}$$

and we denote by co$D^*\phi(x)$ the convex hull of $D^*\phi(x)$. We observe that if ϕ is nondecreasing in a neighborhood of x then all elements of $D^*\phi(x)$ are nonnegative.

Definition 2.3. We say that $\phi : (\underline{x}, +\infty) \to \mathbb{R}$ is semiconcave if there exists a constant $C \geq 0$ such that for all $x, z \in (\underline{x}, +\infty)$ we have $\frac{1}{2}\phi(x) + \frac{1}{2}\phi(z) \leq \phi\left(\frac{x+z}{2}\right) + \frac{1}{4}C|x-z|^2$.

It is obvious that any concave function is also semiconcave.

Lemma 2.4. *([5, Proposition 4.7]) Let ϕ be a semiconcave function in $(\underline{x}, +\infty)$. Then $D^+\phi(x) =$ co$D^*\phi(x)$ for all $x \in (\underline{x}, +\infty)$. Moreover, if $D^+\phi(x)$ is a singleton then ϕ is differentiable at x.*

The following result is a direct consequence of [36, Theorem 25.7] and used in [35]. It allows one to prove the continuous dependence of the consumption and saving policies upon the interest rate r by studying the convergence of the value functions, see also [2] and [10].

Lemma 2.5. *Let $v^{(\iota)}$, $\iota \in \mathbb{N}$ be a sequence of strictly concave, differentiable and bounded functions defined on $(\underline{x}, +\infty)$. If $v^{(\iota)}(x) \to v(x)$ for all x as $\iota \to +\infty$, then $Dv^{(\iota)}$ converges to Dv locally uniformly.*

3 Analysis of the Hamilton-Jacobi-Bellman equation

Let us recall the definition of viscosity solution for the HJB system (2.2), which extends naturally the definition used in [10] when the utility is of CRRA type.

Definition 3.1.

1. An upper semicontinuous (u.s.c.) function $v = (v_1, v_2)$ is said to be a viscosity subsolution of (2.2) at x, if whenever φ is a smooth function and $v_j - \varphi$ has a local maximum at x, then

$$\frac{\rho}{\theta} v_j(x) \leq H(x, y_j, v_j(x), D\varphi(x)) + \lambda_j(v_{\bar{j}}(x) - v_j(x)).$$

2. A lower semicontinuous (l.s.c.) function $v = (v_1, v_2)$ is said to be a viscosity supersolution of (2.2) at x, if whenever φ is a smooth function and $v_j - \varphi$ has a local minimum at x, then

$$\frac{\rho}{\theta} v_j(x) \geq H(x, y_j, v_j(x), D\varphi(x)) + \lambda_j(v_{\bar{j}}(x) - v_j(x)).$$

A continuous function v is said to be a constrained viscosity solution to system (2.2) if v is a viscosity supersolution in $(\underline{x}, \infty)$ and a viscosity subsolution in $[\underline{x}, \infty)$.

It is useful to state the following equivalent definition using sub- and superdifferentials.

Definition 3.2.

1. An upper semicontinuous (u.s.c.) function $v = (v_1, v_2)$ is a viscosity subsolution of (2.2) at x, if
$$\frac{\rho}{\theta} v_j(x) \leq H(x, y_j, v_j(x), p) + \lambda_j (v_{\bar{j}}(x) - v_j(x)) \quad \forall p \in D^+ v_j(x).$$
2. A lower semicontinuous (l.s.c.) function $v = (v_1, v_2)$ is a viscosity supersolution of (2.2) at x, if
$$\frac{\rho}{\theta} v_j(x) \geq H(x, y_j, v_j(x), p) + \lambda_j (v_{\bar{j}}(x) - v_j(x)) \quad \forall p \in D^- v_j(x).$$

We next state a strong comparison principle for viscosity sub- and supersolutions of (2.2).

Proposition 3.3. *Assume that* $\mathsf{u} = (\mathsf{u}_1, \mathsf{u}_2)$ *and* $\mathsf{v} = (\mathsf{v}_1, \mathsf{v}_2)$ *are bounded viscosity sub- and supersolution of system* (2.2)*. We extend* v_j *at* $\underline{x}$ *by setting* $\mathsf{v}_j(\underline{x}) = \liminf_{z \to \underline{x},\, z > \underline{x}} \mathsf{v}_j(z)$*. Then* $\mathsf{u} \leq \mathsf{v}$ *in* $[\underline{x}, +\infty)$*, i.e.* $\mathsf{u}_j \leq \mathsf{v}_j$ *in* $[\underline{x}, +\infty)$ *for* $j = 1, 2$*.*

The proof is contained in appendix A. Note that this result holds also if $\gamma\psi = 1$, cf. [10].

Next we obtain some explicit sub- and supersolution of system (2.2). To gain intuition, we first consider the decoupled system, i.e. the case when $\lambda_j = 0$, $j \in \{1, 2\}$:
$$\frac{\rho}{\theta} v_j(x) = H(x, y_j, v_j, Dv_j). \tag{3.1}$$

Lemma 3.4. *Consider*
$$\check{\mathsf{u}}_j = \frac{(rx + y_j)^{1-\gamma}}{1-\gamma}, \quad \check{\mathsf{v}}_j = \frac{(b(x + y_j/r))^{1-\gamma}}{1-\gamma}, \tag{3.2}$$
where b is defined in Table 1. The functions $\check{\mathsf{u}}_j$ *and* $\check{\mathsf{v}}_j$ *are respectively a subsolution of* (3.1) *in* $[\underline{x}, +\infty)$ *and a supersolution of* (3.1) *in* $(\underline{x}, +\infty)$*.*

Proof. $\check{\mathsf{v}}_j(x)$ is a classical solution to (3.1) in $(\underline{x}, +\infty)$. Hence $\check{\mathsf{v}}_j$ is a supersolution of (3.1) in $(\underline{x}, +\infty)$.

It is straightforward to verify that for all $x > \underline{x}$, $\check{\mathsf{u}}_j$ satisfies
$$\frac{\rho}{\theta} \check{\mathsf{u}}_j \leq H(x, y_j, \check{\mathsf{u}}_j, D\check{\mathsf{u}}_j).$$
Let φ be a smooth function such that $\check{\mathsf{u}}_j - \varphi$ has a local maximum at $\underline{x}$. It follows from (2.8) that
$$H(\underline{x}, y_j, \check{\mathsf{u}}_j(\underline{x}), D\varphi(\underline{x})) \geq \rho \frac{(r\underline{x} + y_j)^{1-\psi^{-1}}}{1-\psi^{-1}} \big((1-\gamma)\check{\mathsf{u}}_j(\underline{x})\big)^{\frac{\psi^{-1}-\gamma}{1-\gamma}} = \rho \frac{(r\underline{x} + y_j)^{1-\gamma}}{1-\psi^{-1}}.$$
On the other hand, we have
$$\frac{\rho}{\theta} \check{\mathsf{u}}_j(\underline{x}) = \rho \frac{1-\gamma}{1-\psi^{-1}} \frac{(r\underline{x} + y_j)^{1-\gamma}}{1-\gamma} = \rho \frac{(r\underline{x} + y_j)^{1-\gamma}}{1-\psi^{-1}},$$
hence $\frac{\rho}{\theta} \check{\mathsf{u}}_j(\underline{x}) \leq H(\underline{x}, y_j, \check{\mathsf{u}}_j(\underline{x}), D\varphi(\underline{x}))$ and $\check{\mathsf{u}}_j$ is a subsolution of (3.1) in $[\underline{x}, +\infty)$. $\square$

We can then build the sub- and supersolutions of (1.4) by taking advantage of Theorem 3.4.

Proposition 3.5. *Assume $r > 0$. The functions*

$$(\check{\mathsf{u}}_1, \check{\mathsf{u}}_2) = \left(\frac{(rx+y_1)^{1-\gamma}}{1-\gamma}, \frac{(rx+y_1)^{1-\gamma}}{1-\gamma} \right) \quad \text{and} \quad (\check{\mathsf{v}}_1, \check{\mathsf{v}}_2) = \left(\frac{(b(x+y_2/r))^{1-\gamma}}{1-\gamma}, \frac{(b(x+y_2/r))^{1-\gamma}}{1-\gamma} \right)$$

are respectively a subsolution of (2.2) *in* $[\underline{x}, +\infty)$ *and a supersolution of* (2.2) *in* $(\underline{x}, +\infty)$.

Proof. We first consider $(\check{\mathsf{u}}_1, \check{\mathsf{u}}_1)$. It is clear from Theorem 3.4 that for $j = 1$,

$$H(x, y_1, \check{\mathsf{u}}_1, D\check{\mathsf{u}}_1) + \lambda_1(\check{\mathsf{u}}_1(x) - \check{\mathsf{u}}_1(x)) \geq \frac{\rho}{\theta}\check{\mathsf{u}}_1(x), \quad \forall x > \underline{x}.$$

For $x = \underline{x}$, the argument is the same as in the proof of Theorem 3.4. For $j = 2$ and $x > \underline{x}$:

$$H(x, y_2, \check{\mathsf{u}}_1, D\check{\mathsf{u}}_1) + \lambda_1(\check{\mathsf{u}}_1(x) - \check{\mathsf{u}}_1(x)) = H(x, y_1, \check{\mathsf{u}}_1, D\check{\mathsf{u}}_1) + \lambda_1(\check{\mathsf{u}}_1(x) - \check{\mathsf{u}}_1(x)) + (y_2 - y_1)D\check{\mathsf{u}}_1(x) \geq \zeta\check{\mathsf{u}}_1(x).$$

Let φ be a smooth function such that $\check{\mathsf{u}}_1 - \varphi$ has a local maximum at $\underline{x}$. The inequality $D\varphi(\underline{x}) \geq D\check{\mathsf{u}}_1(\underline{x}) > 0$ implies

$$H(\underline{x}, y_2, \check{\mathsf{u}}_1(\underline{x}), D\varphi(\underline{x})) + \lambda_1(\check{\mathsf{u}}_1(\underline{x}) - \check{\mathsf{u}}_1(\underline{x})) > \zeta\check{\mathsf{u}}_1(\underline{x}).$$

We argue similarly with $(\check{\mathsf{v}}_2, \check{\mathsf{v}}_2)$ and notice that for the supersolution property we only need to consider $x > \underline{x}$. □

Remark 3.6. *Next we use Theorem 3.1 and Theorem 2.1 to show why $\check{\mathsf{v}}_j$ is not a subsolution to* (3.1)*, even though it is a classical solution in $(\underline{x}, +\infty)$. Note that $D\check{\mathsf{v}}_j(\underline{x}) = b^{1-\gamma}(\underline{x} + y_j/r)^{-\gamma}$, we consider the test function*

$$\varphi(x) = \rho b^{\frac{1}{\psi}-\gamma} r^{-1/\psi} \frac{(x + y_j/r)^{1-\gamma}}{1-\gamma}, \qquad D\varphi(x) = \rho b^{\frac{1}{\psi}-\gamma} r^{-1/\psi}(x + y_j/r)^{-\gamma}.$$

Since $b > r$, $\rho > r$ and $\psi > 0$ we have first $r^{-1/\psi} > b^{-1/\psi}$, then $\rho b^{\frac{1}{\psi}-\gamma} r^{-1/\psi} > b^{1+\frac{1}{\psi}-\gamma} b^{-1/\psi} > b^{1-\gamma}$. This implies $D\varphi(\underline{x}) > D\check{\mathsf{v}}_j(\underline{x})$. Therefore $\underline{x}$ is a local maximum of $\check{\mathsf{v}}_j - \varphi$. Now

$$H(\underline{x}, y_j, \check{\mathsf{v}}_j(\underline{x}), D\varphi(\underline{x})) = \frac{\rho b^{\frac{1}{\psi}-\gamma} r^{1-\frac{1}{\psi}} (\underline{x} + y_j/r)^{1-\gamma}}{1 - \frac{1}{\psi}}.$$

Because $\frac{z^{1-\frac{1}{\psi}}}{1-\frac{1}{\psi}}$ is an increasing function in $(0, +\infty)$ and $b > r$, we have

$$\frac{\rho b^{1-\gamma}(\underline{x} + y_j/r)^{1-\gamma}}{1 - \frac{1}{\psi}} > \frac{\rho b^{\frac{1}{\psi}-\gamma} r^{1-\frac{1}{\psi}} (\underline{x} + y_j/r)^{1-\gamma}}{1 - \frac{1}{\psi}}.$$

On the other hand,

$$\frac{\rho b^{1-\gamma}(\underline{x} + y_j/r)^{1-\gamma}}{1 - \frac{1}{\psi}} = \frac{\rho b^{1-\gamma}(\underline{x} + y_j/r)^{1-\gamma}}{\theta(1-\gamma)} = \frac{\rho}{\theta}\check{\mathsf{v}}_j(\underline{x}),$$

hence we have obtained: $\frac{\rho}{\theta}\check{\mathsf{v}}_j(\underline{x}) > H(\underline{x}, y_j, \mathsf{v}_j(\underline{x}), D\varphi(\underline{x}))$. Therefore, $\check{\mathsf{v}}_j$ is not a subsolution at $\underline{x}$.

Similar explicit calculations show that $\check{\mathsf{u}}_j < \check{\mathsf{v}}_j$, in agreement with the comparison principle in Theorem 3.3.

Proposition 3.7. *Assume* $0 < r \leq \rho$. *There exists a unique bounded viscosity solution* $v = (v_1, v_2)$ *to* (2.2)*.*

Proof. We consider the following regularized problem. Given a constant $R > r\underline{x} + y_2$, set

$$H_R(x, y_j, v_j, Dv_j) = \max_{0 \leq c \leq R} \left\{ \mathcal{F}(c, v_j) + (rx + y_j - c)Dv_j \right\},$$

and consider the system

$$\frac{\rho}{\theta} v_j(x) = H_R(x, y_j, v_j, Dv_j) + \lambda_j(v_{\bar{j}}(x) - v_j(x)), \tag{3.3}$$

with state constraints. We proceed in several steps.
Step 1. We claim that for each R, the comparison principle is satisfied for the sub- and supersolution of (3.3). The proof is similar to that of Theorem 3.3 that guarantees the uniqueness of solution if it exists. We observe that

$$\max_{0 \leq c \leq R} \left\{ \mathcal{F}(c, \check{\mathsf{v}}_j) + (rx + y_j - c)D\check{\mathsf{v}}_j \right\} \leq \max_{c \geq 0} \left\{ \mathcal{F}(c, \check{\mathsf{v}}_j) + (rx + y_j - c)D\check{\mathsf{v}}_j \right\},$$

hence $(\check{\mathsf{v}}_1, \check{\mathsf{v}}_2)$ is a supersolution of (3.3). On the other hand, by defining

$$\check{\mathsf{u}}_1^R(x) = \check{\mathsf{u}}_2^R(x) = \begin{cases} \check{\mathsf{u}}_1(x) & \text{if} \quad rx + y_1 \leq R, \\ \dfrac{R^{1-\gamma}}{1-\gamma} & \text{if} \quad rx + y_1 > R, \end{cases}$$

we can use a similar argument as for Theorem 3.5 to show that $(\check{\mathsf{u}}_1^R(x), \check{\mathsf{u}}_2^R(x))$ is a subsolution of (3.3). From the comparison principle, we deduce that for a state constrained viscosity solution $\left(v_1^R, v_2^R\right)$ of (3.3), $\check{\mathsf{u}}_j^R(x) \leq v_j^R(x) \leq \check{\mathsf{v}}_j(x)$ holds for all $x \geq \underline{x}$. The existence of a solution $\left(v_1^R, v_2^R\right)$ to (3.3) follows from Perron's method. We observe that $\left(v_1^R, v_2^R\right)$ is the value function of a stochastic optimal control problem and that v_j^R is concave. Following the same argument as in Theorem 3.10 below, v_j^R belongs to $C^1(\underline{x}, +\infty)$.
Step 2. We now show that $Dv_j^R \geq 0$. Suppose that there exists x^* such that $Dv_j^R(x^*) = \eta < 0$, then for all $x \geq x^*$, $Dv_j^R(x) \leq \eta$ and

$$\max_{0 \leq c \leq R} \left\{ \mathcal{F}(c, v_j^R) + (rx + y_j - c)Dv_j^R \right\} = \mathcal{F}(R, v_j^R) + (rx + y_j - R)Dv_j^R.$$

From (2.5) and $\check{\mathsf{u}}_j(\underline{x}) \leq v_j^R(x)$, we deduce $\mathcal{F}(R, v_j^R) \leq \mathcal{F}(R, \check{\mathsf{u}}_j(\underline{x}))$. Since $Dv_j^R(x) \leq \eta$ for all $x \geq x^*$, $(rx + y_j - R)Dv_j^R \to -\infty$ as $x \to +\infty$. Therefore, $H_R\left(x, y_j, v_j^R, Dv_j^R\right) \to -\infty$ as $x \to +\infty$, which contradicts the boundedness of v_j^R. We have proved that there does not exist x^* such that $Dv_j^R(x^*) < 0$.

Similarly, from (2.5) and $R > r\underline{x} + y_2$, and the state constraint at $\underline{x}$,

$$H_R\left(\underline{x}, y_j, v_j^R(\underline{x}), Dv_j^R(\underline{x})\right) = H\left(\underline{x}, y_j, v_j^R(\underline{x}), Dv_j^R(\underline{x})\right) \geq H\left(\underline{x}, y_j, \check{\mathsf{v}}_j(\underline{x}), Dv_j^R(\underline{x})\right),$$

we deduce from the coercivity of H that $Dv_j^R(\underline{x})$ is bounded independently of R. The upper bound on Dv_j^R then follows from the concavity of v_j^R.
Step 3. Suppose $R' > R$, we deduce from

$$H_R\left(x, y_j, v_j^R, Dv_j^R\right) \leq H_{R'}\left(x, y_j, v_j^R, Dv_j^R\right),$$

that $\left(v_1^R, v_2^R\right)$ is a subsolution of the system

$$\frac{\rho}{\theta} v_j(x) = H_{R'}(x, y_j, v_j, Dv_j) + \lambda_j(v_{\bar{\jmath}}(x) - v_j(x)). \tag{3.4}$$

The comparison principle then gives $v_j^R \leq v_j^{R'}$. Therefore, v_j^R depends on R in a monotone increasing manner for R sufficiently large. Since $\left(v_1^R, v_2^R\right)$ is bounded above by the supersolution $(\check{\mathsf{v}}_1, \check{\mathsf{v}}_2)$ independently of R, we infer that $\left(v_1^R, v_2^R\right)$ converges pointwise to the limit (v_1, v_2) as $R \to +\infty$. By the uniform boundedness of Dv_j^R and the ArzelàAscoli theorem, the sequence $\left(v_1^R, v_2^R\right)$ converges locally uniformly to (v_1, v_2). Moreover the functions v_j are continuous, strictly increasing, concave and tend to 0 as $x \to \infty$. From the stability of viscosity solutions, we conclude that (v_1, v_2) is a constrained viscosity solution to (2.2), in fact the unique one from the comparison principle. ☐

Proposition 3.8. *The solution v_j is locally Lipschitz.*

Proof. We consider some $x^* > \underline{x}$ and its neighborhood $(x^* - \varepsilon, x^* + \varepsilon)$ such that $x^* - \varepsilon > \underline{x}$. We consider an interval $[x_1, x_2]$ such that $\underline{x} < x_1 < x^* - \varepsilon < x^* + \varepsilon < x_2 < +\infty$ and then proceed in two steps.
Step 1. We show that there exists $\hat{C} > 0$, $\hat{C}$ depends on x_1, x_2 such that

$$\frac{\rho}{\theta} v_j(x) < H(x, y_j, v_j(x), \hat{C}) + \lambda_j(v_{\bar{\jmath}}(x) - v_j(x)) \quad \forall x \in [x_1, x_2]. \tag{3.5}$$

We observe from the comparison principle and (2.10) that for all $\hat{C} > 0$,

$$H(x_1, y_1, \check{\mathsf{v}}_2(x_2), \hat{C}) \leq H(x, y_j, v_j(x), \hat{C}),$$

and

$$\frac{\rho}{\theta} v_j(x) - \lambda_j(v_{\bar{\jmath}}(x) - v_j(x)) < \lambda_j \check{\mathsf{u}}_1(x_1).$$

Moreover, $H(x_1, y_1, \check{\mathsf{v}}_2(x_2), C)$ is a monotone increasing function of C if

$$C > \rho(rx_1 + y_1)^{-\psi^{-1}}\left((1-\gamma)\check{\mathsf{v}}_2(x_2)\right)^{\frac{\psi^{-1}-\gamma}{1-\gamma}}.$$

Therefore, there exists $\hat{C} > 0$ such that $\lambda_j \check{\mathsf{u}}_1(x_1) < H(x_1, y_1, \check{\mathsf{v}}_2(x_2), \hat{C})$, hence (3.5) also holds.
Step 2. We now consider the minimization problem, for some $C_0 > \hat{C}$, C_0 may depend on x^* and ε, and $x \in (x^* - \varepsilon, x^* + \varepsilon)$,

$$\min_{z \in [x_1, x_2]} v_j(z) + C_0|z - x|. \tag{3.6}$$

From the boundedness of v_j, C_0 can be chosen large enough such that neither x_1 nor x_2 can be the minimizer of (3.6). We claim that the unique minimizer in (3.6) is $\hat{z} = x$. Otherwise, if $\hat{z} \neq x$ then $|z - x|$ is differentiable at $\hat{z}$ and, from the inequality for supersolution in Theorem 3.1 it follows

$$\frac{\rho}{\theta} v_j(\hat{z}) \geq H(\hat{z}, y_j, v_j(\hat{z}), -C_0 \frac{\hat{z} - x}{|\hat{z} - x|}) + \lambda_j(v_{\bar{\jmath}}(\hat{z}) - v_j(\hat{z})).$$

This is in contradiction with (3.5) if $\hat{z} < x$, or with (2.4) if $\hat{z} > x$. Therefore, $v_j(z) + C_0|z - x| \geq v_j(x)$ for all $z \in (x^* - \varepsilon, x^* + \varepsilon)$. By symmetry we can show $v_j(x) + C_0|x - z| \geq v_j(z)$. ☐

Proposition 3.9. *The viscosity solution $v = (v_1, v_2)$ is the value function of the optimal control problem* (1.1)*. Furthermore, the function $v_j(x)$ is strictly concave in x.*

Proof. We consider the optimal control problem, involving the function v found in Theorem 3.8:

$$\mathbf{V}(x,y) = \max_{c_\tau} \mathbb{E}\left[\int_t^\infty f(c_\tau, v_\tau)\, d\tau \big| \boldsymbol{x}_t = x,\, \boldsymbol{y}_t = y\right], \quad \left\{ \begin{array}{rcl} d\boldsymbol{x}_\tau & = & (r\boldsymbol{x}_\tau + \boldsymbol{y}_\tau - c_\tau)dt, \ \tau > t, \\ \boldsymbol{x}_\tau & \geq & \underline{x}. \end{array} \right. \tag{3.7}$$

The value function of (1.1) is a fixed point of (3.7). From the dynamic programming principle, the value function $\mathbf{V}$ of (3.7) is a viscosity solution of :

$$\frac{\rho}{\theta}\mathbf{V}_j(x) = H(x, y_j, v_j, D\mathbf{V}_j) + \lambda_j(\mathbf{V}_{\bar{j}}(x) - \mathbf{V}_j(x)). \tag{3.8}$$

From the local Lipschitz continuity of v, we can then show the viscosity solution to (3.8) is unique. Moreover, from Theorem 3.8, we know $\mathbf{V} = v$ is itself a viscosity solution of (3.8). Therefore, the function v is a solution of (1.1).

From (2.6), $f(c, v_j)$ is concave in (c, v_j). The concavity of the value function then follows from [16]. □

Proposition 3.10. *The function v_j belongs to $C^1(\underline{x}, +\infty)$.*

Proof. Our strategy is similar to that in [5, Section 5.2, Proposition 5.7]. The function v_j is concave and therefore semiconcave. Since v_j is differentiable almost everywhere in $(\underline{x}, +\infty)$, the set $D^*v_j(x)$ is nonempty and closed for any $x > \underline{x}$.

To prove that v_j is differentiable in $(\underline{x}, +\infty)$, we only need to establish that D^+v_j is a singleton. From the *semiconcavity* of v_j, $D^+v_j = \mathrm{co}D^*v_j$ and we only need to prove that D^*v_j is a singleton.

Suppose by contradiction $p^1 \neq p^2 \in D^*v_j$. Then there exist subsequences $\{x_n\}$, $\{z_k\}$ such that v_j is differentiable at each $\{x_n\}$ and $\{z_k\}$, and

$$x = \lim_{n\to+\infty} x_n = \lim_{k\to+\infty} z_k, \quad p^1 = \lim_{n\to+\infty} Dv_j(x_n), \quad p^2 = \lim_{k\to+\infty} Dv_j(z_k).$$

$$\begin{aligned} \frac{\rho}{\theta}v_j(x_n) - H(x_n, y_j, v_j(x_n), Dv_j(x_n)) &= \lambda_j(v_{\bar{j}}(x_n) - v_j(x_n)), \\ \frac{\rho}{\theta}v_j(z_k) - H(z_k, y_j, v_j(z_k), Dv_j(z_k)) &= \lambda_j(v_{\bar{j}}(z_k) - v_j(z_k)). \end{aligned} \tag{3.9}$$

Using the supersolution found in Theorem 3.4 and the comparison principle in Theorem 3.3, we deduce that $v_j(x)$ is bounded away from 0 for any $x < +\infty$. This allows us to take advantage of the continuity properties of H and the continuity of v_j, $j \in \{1, 2\}$ to infer that

$$\frac{\rho}{\theta}v_j(x) - H\left(x, y_j, v_j(x), p^1\right) = \frac{\rho}{\theta}v_j(x) - H\left(x, y_j, v_j(x), p^2\right) = \lambda_j(v_{\bar{j}}(x) - v_j(x)). \tag{3.10}$$

Set $\bar{p} = (p^1 + p^2)/2$, from the strict convexity of H we infer

$$\frac{\rho}{\theta}v_j(x) - H\left(x, y_j, v_j(x), \bar{p}\right) > \frac{\rho}{\theta}v_j(x) - \frac{1}{2}H\left(x, y_j, v_j(x), p^1\right) - \frac{1}{2}H\left(x, y_j, v_j(x), p^2\right) = \lambda_j(v_{\bar{j}}(x) - v_j(x)). \tag{3.11}$$

On the other hand $\bar{p} \in \mathrm{co}D^*v_j$, hence from Theorem 2.4 $\bar{p} \in D^+v_j$. Therefore, by Definition 3.2

$$\frac{\rho}{\theta}v_j(x) - H\left(x, y_j, v_j(x), \bar{p}\right) \leq \lambda_j(v_{\bar{j}}(x) - v_j(x)),$$

in contradiction with (3.11), therefore $D^+ v_j = \mathrm{co} D^* v_j$ is a singleton.

Finally, Dv_j is continuous in $(\underline{x}, +\infty)$, by the upper semicontinuity of the multivalued map $D^+ v_j$ for the semiconcave function v_j. The coercivity of the Hamiltonian and the concavity of v_j imply that Dv_j is uniformly bounded in $(\underline{x}, +\infty)$. ☐

Proposition 3.11. *The function Dv_j is uniformly continuous in $[\underline{x}, R]$ for any $R > \underline{x}$. Moreover, Dv_j is bounded on $[\underline{x}, R]$ by a constant depending only on γ, ψ, (y_1, y_2) and (λ_1, λ_2).*

Proof. From the concavity of v_j, $Dv_j(x)$ is monotone increasing as $x \to \underline{x}$. There exists $Dv_j(\underline{x}^+)$ such that $Dv_j(\underline{x}^+) = \lim_{x \to \underline{x}, x > \underline{x}} Dv_j(x)$. Moreover, we define $Dv_j(\underline{x}) = \lim_{\epsilon \to 0, \epsilon > 0} \frac{v_j(\underline{x}+\epsilon) - v_j(\underline{x})}{\epsilon}$ if it exists. Since v_j is C^1 in $(\underline{x}, R]$ and continuous in $[\underline{x}, R]$, we obtain that $Dv_j(\underline{x})$ exists and $Dv_j(\underline{x}) = Dv_j(\underline{x}^+)$ by the mean value theorem. Therefore, Dv_j is continuous on the interval $[\underline{x}, R]$ and we obtain the uniform continuity by Heine-Borel theorem. ☐

Corollary 3.12. *Assume $0 < r \leq \rho$. There exists a constant $c_{\min} > 0$, independent of r, such that $c_j(x) > c_{\min}$ for all $x \geq \underline{x}$ and $j \in \{1, 2\}$.*

Proof. From (2.7) and Theorem 3.11, the first order condition for the optimal control (2.7) holds pointwise. Theorem 3.8 and Theorem 3.11 imply that $(Dv_j(x))^{-\psi}$ is bounded below for all $x \geq \underline{x}$. By comparison with the subsolution $\check{\mathsf{u}}_1$, see Theorem 3.5, we obtain that

$$((1-\gamma) v_j(x))^{\frac{1-\gamma\psi}{1-\gamma}} \geq ((1-\gamma)\check{\mathsf{u}}_1(x))^{\frac{1-\gamma\psi}{1-\gamma}} \geq (rx + y_1)^{1-\gamma\psi}.$$

If $\underline{x} \geq 0$, then $(rx + y_1)^{1-\gamma\psi} \geq y_1^{1-\gamma\psi}$. If $\underline{x} < 0$, then $(rx + y_1)^{1-\gamma\psi} \geq (\rho\underline{x} + y_1)^{1-\gamma\psi}$. Therefore, we can find a lower bound $c_{\min}$ uniform w.r.t. r. ☐

Now we can consider the continuous dependence of the consumption and saving policies upon the interest rate r.

Proposition 3.13. *We denote by $v^{(\iota)}$ the solution to system (1.4) corresponding to an interest rate $r^{(\iota)}$ with $0 \leq r^{(\iota)} \leq \rho$. For $r^{(\iota)} \to r$, the sequence $v^{(\iota)}$ converges in $C^1[\underline{x}, R]$ to v for any $R > \underline{x}$.*

Proof. From (2.10), Theorem 3.5 and Theorem 3.8, $H_v(x, y_j, v_j^{(\iota)}, Dv_j^{(\iota)})$ is uniformly bounded for all $r^{(\iota)}$ such that $0 \leq r^{(\iota)} \leq \rho$. By the stability of constrained viscosity solution $v^{(\iota)}$ converges to v uniformly. Since $v_j^{(\iota)}$ is strictly concave (Theorem 3.9), the local uniform convergence of $Dv_j^{(\iota)}$ to Dv_j follows from Theorem 2.5. ☐

We denote by $c_j^{(\iota)}$ and $s_j^{(\iota)}$ the consumption and saving policies with interest rate $r^{(\iota)}$. From Theorem 3.13, the sequences $c_j^{(\iota)}$ and $s_j^{(\iota)}$ converge locally uniformly to c_j and s_j as $r^{(\iota)} \to r$.

The semiconvexity of the value function can be obtained with control theoretic arguments, as in [11].

Proposition 3.14. *The value function (v_1, v_2) of (1.1) is locally semiconvex.*

Proof. From the stationarity of the control problem (1.1), we can take $t = 0$ in (1.1) and for the rest of the proof, we denote by t the generic time variable. There exists T such that the optimal trajectory $\boldsymbol{x}_t^*$ starting from (x, y) does not reach $\underline{x}$ for all $y \in \{y_1, y_2\}$ and all realizations of $\boldsymbol{y}_t$. From Theorem 3.12, for all $t \geq 0$:

$$c\left(\boldsymbol{x}_t^*, \boldsymbol{y}_t\right) = \rho^{\psi} \left(Dv\left(\boldsymbol{x}_t^*, \boldsymbol{y}_t\right)\right)^{-\psi} \left((1-\gamma) v\left(\boldsymbol{x}_t^*, \boldsymbol{y}_t\right)\right)^{\frac{1-\gamma\psi}{1-\gamma}} > c_{\min}.$$

For $0 < h < x - \underline{x}$ and ε that will be chosen soon, consider the trajectories

$$\begin{cases} d\boldsymbol{x}_t^+ = r\boldsymbol{x}_t^+ + \boldsymbol{y}_t - c\left(\boldsymbol{x}_t^*, \boldsymbol{y}_t\right) - \varepsilon e^{rt}, & \boldsymbol{x}_0^+ = x + h, \\ d\boldsymbol{x}_t^- = r\boldsymbol{x}_t^- + \boldsymbol{y}_t - c\left(\boldsymbol{x}_t^*, \boldsymbol{y}_t\right) + \varepsilon e^{rt}, & \boldsymbol{x}_0^- = x - h. \end{cases} \tag{3.12}$$

We have

$$\begin{aligned} d\left(\boldsymbol{x}_t^+ - \boldsymbol{x}_t^*\right) &= r\left(\boldsymbol{x}_t^+ - \boldsymbol{x}_t^*\right) - \varepsilon e^{rt}, \\ d\left(\boldsymbol{x}_t^- - \boldsymbol{x}_t^*\right) &= r\left(\boldsymbol{x}_t^- - \boldsymbol{x}_t^*\right) + \varepsilon e^{rt}, \end{aligned}$$

and by taking $\varepsilon = 2h/T$ we obtain

$$\begin{aligned} \boldsymbol{x}_t^+ - \boldsymbol{x}_t^* &= e^{rt}\left(\boldsymbol{x}_0^+ - x\right) - \varepsilon \int_0^t e^{r(t-\tau)} e^{r\tau} d\tau = he^{rt} - \varepsilon t e^{rt} = he^{rt}\left(1 - \frac{2t}{T}\right), \\ \boldsymbol{x}_t^- - \boldsymbol{x}_t^* &= -he^{rt} + \varepsilon t e^{rt} = -he^{rt}\left(1 - \frac{2t}{T}\right). \end{aligned}$$

By construction, $\boldsymbol{x}_{T/2}^+ = \boldsymbol{x}_{T/2}^- = \boldsymbol{x}_{T/2}^*$. Moreover,

$$\boldsymbol{x}_t^- < \boldsymbol{x}_t < \boldsymbol{x}_t^+ \text{ and } \frac{\boldsymbol{x}_t^+ - \boldsymbol{x}_t^-}{2} = he^{rt}\left(1 - \frac{2t}{T}\right), \quad \forall t < T/2. \tag{3.13}$$

Consider the positive value δ, depending on x:

$$\delta := \min_{\tau \in [0, T/2]} x^*(\tau) > 0, \tag{3.14}$$

and choose h such that

$$\max_{t \in [0, T/2]} he^{rt}\left(1 - \frac{2t}{T}\right) < \delta \quad \text{and} \quad c_{\min} - \frac{2he^{rT}}{T} > \frac{c_{\min}}{2}. \tag{3.15}$$

Without loss of generality we take $y_0 = y_j$. From the dynamic programming principle

$$\begin{aligned} v_j(x) &= \mathbb{E}\left[\int_0^{T/2} f\left(c\left(\boldsymbol{x}_t^*, \boldsymbol{y}_t\right), v\left(\boldsymbol{x}_t^*, \boldsymbol{y}_t\right)\right) dt + v(\boldsymbol{x}_{T/2}^*, \boldsymbol{y}_{T/2})\right], \\ v_j(x+h) &= \mathbb{E}\left[\int_0^{T/2} f\left(c\left(\boldsymbol{x}_t^*, \boldsymbol{y}_t\right) - \varepsilon e^{rt}, v\left(\boldsymbol{x}_t^+, \boldsymbol{y}_t\right)\right) dt + v(\boldsymbol{x}_{T/2}^*, \boldsymbol{y}_{T/2})\right], \\ v_j(x-h) &= \mathbb{E}\left[\int_0^{T/2} f\left(c\left(\boldsymbol{x}_t^*, \boldsymbol{y}_t\right) + \varepsilon e^{rt}, v\left(\boldsymbol{x}_t^-, \boldsymbol{y}_t\right)\right) dt + v(\boldsymbol{x}_{T/2}^*, \boldsymbol{y}_{T/2})\right]. \end{aligned}$$

Using the notation from (2.3), we have

$$\begin{aligned} & v_j(x+h) - 2v_j(x) + v_j(x-h) \\ &= \mathbb{E}\int_0^{T/2} \left(\mathcal{F}\left(c\left(\boldsymbol{x}_t^*, \boldsymbol{y}_t\right) - \varepsilon e^{rt}, v\left(\boldsymbol{x}_t^+, \boldsymbol{y}_t\right)\right) - 2\mathcal{F}\left(c\left(\boldsymbol{x}_t^*, \boldsymbol{y}_t\right), v\left(\boldsymbol{x}_t^*, \boldsymbol{y}_t\right)\right) + \mathcal{F}\left(c\left(\boldsymbol{x}_t^*, \boldsymbol{y}_t\right) + \varepsilon e^{rt}, v\left(\boldsymbol{x}_t^-, \boldsymbol{y}_t\right)\right)\right) dt \\ &\quad - \frac{\rho}{\theta}\mathbb{E}\int_0^{T/2} \left(v\left(\boldsymbol{x}_t^+, \boldsymbol{y}_t\right) - 2v\left(\boldsymbol{x}_t^*, \boldsymbol{y}_t\right) + v\left(\boldsymbol{x}_t^-, \boldsymbol{y}_t\right)\right) dt \\ &> \mathbb{E}\int_0^{T/2} \underbrace{\left(\mathcal{F}\left(c\left(\boldsymbol{x}_t^*, \boldsymbol{y}_t\right) - \varepsilon e^{rt}, v\left(\boldsymbol{x}_t^+, \boldsymbol{y}_t\right)\right) - 2\mathcal{F}\left(c\left(\boldsymbol{x}_t^*, \boldsymbol{y}_t\right), v\left(\boldsymbol{x}_t^*, \boldsymbol{y}_t\right)\right) + \mathcal{F}\left(c\left(\boldsymbol{x}_t^*, \boldsymbol{y}_t\right) + \varepsilon e^{rt}, v\left(\boldsymbol{x}_t^-, \boldsymbol{y}_t\right)\right)\right)}_{(I)} dt, \end{aligned} \tag{3.16}$$

where for the last inequality we used (3.13) and the concavity of v.

Next, by using again the concavity of v and $\boldsymbol{x}_t^* = (\boldsymbol{x}_t^+ + \boldsymbol{x}_t^-)/2$,

$$v\left(\boldsymbol{x}_t^*, \boldsymbol{y}_t\right) = v\left((\boldsymbol{x}_t^+ + \boldsymbol{x}_t^-)/2, \boldsymbol{y}_t\right) > \frac{v\left(\boldsymbol{x}_t^+, \boldsymbol{y}_t\right) + v\left(\boldsymbol{x}_t^-, \boldsymbol{y}_t\right)}{2}.$$

From (2.5) $-\mathcal{F}(c, v)$ is increasing in v, we have

$$-\mathcal{F}\left(c\left(\boldsymbol{x}_t^*, \boldsymbol{y}_t\right), v\left(\boldsymbol{x}_t^*, \boldsymbol{y}_t\right)\right) > -\mathcal{F}\left(c\left(\boldsymbol{x}_t^*, \boldsymbol{y}_t\right), \frac{v\left(\boldsymbol{x}_t^+, \boldsymbol{y}_t\right) + v\left(\boldsymbol{x}_t^-, \boldsymbol{y}_t\right)}{2}\right).$$

$$\begin{aligned}(I) > \mathcal{F}\left(c\left(\boldsymbol{x}_t^*, \boldsymbol{y}_t\right) - \varepsilon e^{rt}, v\left(\boldsymbol{x}_t^+, \boldsymbol{y}_t\right)\right) - 2\mathcal{F}\left(c\left(\boldsymbol{x}_t^*, \boldsymbol{y}_t\right), \frac{v\left(\boldsymbol{x}_t^+, \boldsymbol{y}_t\right) + v\left(\boldsymbol{x}_t^-, \boldsymbol{y}_t\right)}{2}\right) \\ + \mathcal{F}\left(c\left(\boldsymbol{x}_t^*, \boldsymbol{y}_t\right) + \varepsilon e^{rt}, v\left(\boldsymbol{x}_t^-, \boldsymbol{y}_t\right)\right).\end{aligned}$$

where $c\left(\boldsymbol{x}_t^*, \boldsymbol{y}_t\right) - \varepsilon e^{rt} > 0$ from the choice of h in (3.15). From (3.15) $x + \delta \geq \boldsymbol{x}_t^+$ with δ defined in (3.14). From the monotonicity of v and by comparison with the supersolution $\check{\mathsf{v}}_2$, we have

$$v\left(\boldsymbol{x}_t^+, \boldsymbol{y}_t\right) \leq v\left(x + \delta, \boldsymbol{y}_t\right) \leq \check{\mathsf{v}}_2(x + \delta).$$

From Theorem 3.12 we obtain that for all

$$c \in [c\left(\boldsymbol{x}_t^*, \boldsymbol{y}_t\right) - \varepsilon e^{rt}, c\left(\boldsymbol{x}_t^*, \boldsymbol{y}_t\right) + \varepsilon e^{rt}], \quad v \in [v\left(\boldsymbol{x}_t^-, \boldsymbol{y}_t\right), v\left(\boldsymbol{x}_t^+, \boldsymbol{y}_t\right)], \tag{3.17}$$

$$\begin{aligned}&\mathcal{F}_{cc}(c, v) \geq \frac{-\rho c_{\min}^{-\psi^{-1}-1}}{\psi((1-\gamma)\check{\mathsf{v}}_2(x+\delta))^{\theta-1}} = C_1, \quad \mathcal{F}_{vv}(c, v) \geq (\gamma - \psi^{-1})\frac{\rho c_{\min}^{1-\psi^{-1}}}{((1-\gamma)\check{\mathsf{v}}_2(x+\delta))^{\theta+1}} = C_2, \\ &0 < \mathcal{F}_{cv}(c, v) \leq (\psi^{-1} - \gamma)\frac{\rho c_{\min}^{-\psi^{-1}}}{((1-\gamma)\check{\mathsf{v}}_2(x+\delta))^{\theta}} = C_3.\end{aligned}$$

There exists $C > \max\{-C_1, -C_2, C_3\}$ such that the term (I) in (3.16) satisfies

$$(I) > -C\left(|\varepsilon e^{rt}|^2 + \left|\frac{v\left(\boldsymbol{x}_t^+, \boldsymbol{y}_t\right) - v\left(\boldsymbol{x}_t^-, \boldsymbol{y}_t\right)}{2}\right|^2\right).$$

Since $\varepsilon = 2h/T$ and $t \leq T/2$ in (3.16) we have $|\varepsilon e^{rt}|^2 = \frac{4e^{2rt}h^2}{T^2} \leq \frac{4e^{\rho T}h^2}{T^2}$. From Theorem 3.11 and (3.13), we have

$$\left|\frac{v\left(\boldsymbol{x}_t^+, \boldsymbol{y}_t\right) - v\left(\boldsymbol{x}_t^-, \boldsymbol{y}_t\right)}{2}\right|^2 \leq \|Dv_j\|_{L^\infty}\left|\frac{\boldsymbol{x}_t^+ - \boldsymbol{x}_t^-}{2}\right|^2 \leq e^{\rho T}\|Dv_j\|_{L^\infty}h^2.$$

Therefore from (3.16) we obtain

$$v_j(x+h) - 2v_j(x) + v_j(x-h) > -\frac{CTe^{\rho T}}{2}\left(\frac{4}{T^2} + \|Dv_j\|_{L^\infty}\right)h^2$$

and we obtain the local semi-convexity of v_j from Theorem 2.3. □

From Theorem 3.9 and Theorem 3.14 v_j is strictly concave and locally semiconvex. Then, from [20, Lemma 10.7]:

Proposition 3.15. *The function v_j is $W^{2,\infty}_{loc}$ in $(\underline{x}, +\infty)$.*

Below, we prove the expected fact that the value of the productive agents is larger than that of the unproductive ones:

Proposition 3.16. *We have $v_2 > v_1$.*

Proof. Let $v_2 - v_1$ achieve its minimum at $\hat{x}$. Assume by contradiction that $v_2(\hat{x}) - v_1(\hat{x}) \le 0$. We distinguish two cases.

Case 1: $\hat{x} > \underline{x}$. From Theorem 3.10, we deduce that $Dv_2(\hat{x}) = Dv_1(\hat{x})$ and

$$\left(\frac{\rho}{\theta} + \lambda_1 + \lambda_2\right) \underbrace{(v_2(\hat{x}) - v_1(\hat{x}))}_{\le 0} = (y_2 - y_1) Dv_2(\hat{x}) + \underbrace{H(\hat{x}, y_1, v_2(\hat{x}), Dv_2(\hat{x})) - H(\hat{x}, y_1, v_1(\hat{x}), Dv_1(\hat{x}))}_{>0},$$

the desired contradiction.

Case 2: $\hat{x} = \underline{x}$. Since $v_2 - v_1$ is C^1 up to the boundary and achieves its minimum at $\underline{x}$, $Dv_2(\underline{x}) \ge Dv_1(\underline{x})$. From the state constraint condition,

$$H_p(\underline{x}, y_1, v_1(\underline{x}), Dv_1(\underline{x})) = s_1(\underline{x}) \ge 0,$$

and, from the convexity of $H(x, y_j, v_j, p)$ in the p-variable and $Dv_2(\underline{x}) \ge Dv_1(\underline{x})$,

$$H(\underline{x}, y_1, v_1(\underline{x}), Dv_2(\underline{x})) - H(\underline{x}, y_1, v_1(\underline{x}), Dv_1(\underline{x})) \ge s_1(\underline{x}) \left(Dv_2(\underline{x}) - Dv_1(\underline{x})\right) \ge 0. \tag{3.18}$$

Subtracting the two HJB equations, we obtain

$$\begin{aligned} &\left(\frac{\rho}{\theta} + \lambda_1 + \lambda_2\right) (v_2(\underline{x}) - v_1(\underline{x})) \\ &= (y_2 - y_1) Dv_2(\hat{x}) + H(\underline{x}, y_1, v_2(\underline{x}), Dv_2(\underline{x})) - H(\underline{x}, y_1, v_1(\underline{x}), Dv_1(\underline{x})), \end{aligned}$$

where

$$\begin{aligned} &H(\underline{x}, y_1, v_2(\underline{x}), Dv_2(\underline{x})) - H(\underline{x}, y_1, v_1(\underline{x}), Dv_1(\underline{x})) \\ = &\underbrace{H(\underline{x}, y_1, v_2(\underline{x}), Dv_2(\underline{x})) - H(\underline{x}, y_1, v_1(\underline{x}), Dv_2(\underline{x}))}_{\ge 0} + \underbrace{H(\underline{x}, y_1, v_1(\underline{x}), Dv_2(\underline{x})) - H(\underline{x}, y_1, v_1(\underline{x}), Dv_1(\underline{x}))}_{\ge 0,\, (3.18)}. \end{aligned}$$

This is in contradiction with the assumption $v_2(\underline{x}) \le v_1(\underline{x})$. ☐

The following proposition states that the savings of the unproductive agents are negative for all values of $x > \underline{x}$. This result is well known in the model involving a CRRA utility but its proof is more difficult in the present case, because one has to handle the dependence of the Hamiltonian (hence the optimal consumption policy away from the borrowing limit, see (2.7)) on the value v_j.

Proposition 3.17. *The optimal saving policy s_1 has the following properties: $s_1(x) < 0$ for all $x > \underline{x}$ and $s_1(\underline{x}) = 0$.*

Proof. We argue by contradiction and suppose $s_1(\hat{x}) \geq 0$ for some $\hat{x} > \underline{x}$. We may first suppose that $s_2(\hat{x}) \neq 0$ and $s_1(\hat{x}) > 0$, this implies that the functions v_j are C^2 in a neighborhood of $\hat{x}$ and that $-\infty < D^2 v_j(\hat{x}) < 0$. Differentiating the HJB equations at $\hat{x}$ leads to

$$\begin{aligned}&(\rho - r)\, Dv_1(\hat{x}) + \lambda_1 \left(Dv_1(\hat{x}) - Dv_2(\hat{x})\right)\\ &= s_1(\hat{x}) D^2 v_1(\hat{x}) + \left(1 - \frac{1}{\theta}\right) \rho Dv_1(\hat{x}) + H_v(\hat{x}, y_1, v_1(\hat{x}), Dv_1(\hat{x})) Dv_1(\hat{x}),\end{aligned} \tag{3.19}$$

and

$$\begin{aligned}&(\rho - r)\, Dv_2(\hat{x}) + \lambda_2 \left(Dv_2(\hat{x}) - Dv_1(\hat{x})\right)\\ &= s_2(\hat{x}) D^2 v_2(\hat{x}) + \left(1 - \frac{1}{\theta}\right) \rho Dv_2(\hat{x}) + H_v(\hat{x}, y_2, v_2(\hat{x}), Dv_2(\hat{x})) Dv_2(\hat{x}).\end{aligned} \tag{3.20}$$

If on the contrary, $s_1(\hat{x}) = 0$ or $s_2(\hat{x}) = 0$, then from Theorem 3.15 we know that the functions Dv_j are differentiable almost everywhere and $D^2 v_j$ are essentially bounded in a neighborhood of $\hat{x}$. Therefore, $s_j(x) D^2 v_j(x)$ has a sense at almost every x in a neighborhood of $\hat{x}$ and we can pass to the limit. On the other hand, from the continuity of v_j and Dv_j with respect to x and the continuity of $H_v(x, y_j, v_j, Dv_j)$ w.r.t. v_j and Dv_j,

$$\lim_{x \to \hat{x}} Dv_j(x) = Dv_j(\hat{x}), \quad \lim_{x \to \hat{x}} H_v(x, y_j, v_j(x), Dv_j(x)) Dv_j(\hat{x}) = H_v(\hat{x}, y_j, v_j(\hat{x}), Dv_j(\hat{x})) Dv_j(\hat{x}).$$

Hence, with the convention $s_j(\hat{x}) D^2 v_j(\hat{x}) = \lim_{x \to \hat{x}} s_j(x) D^2 v_j(x)$, we can still write (3.19) and (3.20) if $s_2(\hat{x}) = 0$ or $s_1(\hat{x}) = 0$.

Since $s_1(\hat{x}) \geq 0$ we deduce

$$\begin{aligned}\rho^{\psi} (Dv_1(\hat{x}))^{-\psi} ((1-\gamma) v_1(\hat{x}))^{\frac{1-\gamma\psi}{1-\gamma}} &= c_1(\hat{x}) \leq r\hat{x} + y_1,\\ (Dv_1(\hat{x}))^{1-\psi} &\geq \rho^{1-\psi} ((1-\gamma) v_1(\hat{x}))^{\frac{(1-\gamma\psi)(1-\psi)}{(1-\gamma)\psi}} (r\hat{x} + y_1)^{1-\frac{1}{\psi}}.\end{aligned}$$

This implies

$$H_v(\hat{x}, y_1, v_1(\hat{x}), Dv_1(\hat{x})) \leq \underbrace{-\rho \left(1 - \frac{1}{\theta}\right) ((1-\gamma) v_1(\hat{x}))^{\frac{\psi^{-1}-1}{1-\gamma}} (r\hat{x} + y_1)^{1-\frac{1}{\psi}}}_{\textit{decreasing in } v_1(\hat{x})} \leq -\rho \left(1 - \frac{1}{\theta}\right). \tag{3.21}$$

For the last inequality we used the comparison with the subsolution in (3.2): $v_1(\hat{x}) \geq \frac{(r\hat{x}+y_1)^{1-\gamma}}{1-\gamma}$. This implies

$$\left(1 - \frac{1}{\theta}\right) \rho Dv_1(\hat{x}) + H_v(\hat{x}, y_1, v_1(\hat{x}), Dv_1(\hat{x})) Dv_1(\hat{x}) \leq 0. \tag{3.22}$$

If $s_1(\hat{x}) > 0$, then $s_1(\hat{x}) D^2 v_1(\hat{x}) \leq 0$. Moreover, since $r \leq \rho$ and $Dv_1 \geq 0$, there holds $(\rho - r)\, Dv_1(\hat{x}) \geq 0$. With $s_1(\hat{x}) D^2 v_1(\hat{x}) \leq 0$, $(\rho - r)\, Dv_1(\hat{x}) \geq 0$ and (3.22), we deduce from (3.19) that:

$$Dv_1(\hat{x}) \leq Dv_2(\hat{x}). \tag{3.23}$$

Then, from $v_1(\hat{x}) < v_2(\hat{x})$ in Theorem 3.16 we infer (notice that $H_v(x, y, v, p)$ in fact does not depend on y):

$$H_v(\hat{x}, y_2, v_2(\hat{x}), Dv_2(\hat{x})) < H_v(\hat{x}, y_1, v_1(\hat{x}), Dv_1(\hat{x})) \leq -\rho \left(1 - \frac{1}{\theta}\right), \tag{3.24}$$

hence

$$\left(1-\frac{1}{\theta}\right)\rho Dv_2(\hat{x})+H_v(\hat{x},y_2,v_2(\hat{x}),Dv_2(\hat{x}))Dv_2(\hat{x})<0.$$

Next, since $\rho \geq r$, $Dv_2(\hat{x})>0$ and $Dv_1(\hat{x})\leq Dv_2(\hat{x})$,

$$\left(\rho-r\right)Dv_2(\hat{x})+\lambda_2\left(Dv_2(\hat{x})-Dv_1(\hat{x})\right)\geq 0.$$

Therefore, from (3.20) and (3.24) we deduce $s_2(\hat{x})D^2v_2(\hat{x})>0$. But we know that $D^2v_2(\hat{x})<0$, so $s_2(\hat{x})<0$. This gives

$$c_2(\hat{x})>r\hat{x}+y_2. \tag{3.25}$$

Recall that the assumption $s_1(\hat{x})\geq 0$ is equivalent to

$$c_1(\hat{x})\leq r\hat{x}+y_1. \tag{3.26}$$

With $c_j(\hat{x})>c_{\min}>0$, we deduce from (3.25) and (3.26) that

$$\frac{c_2(\hat{x})}{c_1(\hat{x})}>\frac{r\hat{x}+y_2}{r\hat{x}+y_1}. \tag{3.27}$$

On the other hand,

$$\frac{c_2(\hat{x})}{c_1(\hat{x})}\underbrace{=}_{(2.7)}\left(\frac{Dv_2(\hat{x})}{Dv_1(\hat{x})}\right)^{-\psi}\left(\frac{(1-\gamma)v_2(\hat{x})}{(1-\gamma)v_1(\hat{x})}\right)^{\frac{1-\gamma\psi}{1-\gamma}}\underbrace{\leq}_{(3.23)}\left(\frac{(1-\gamma)v_2(\hat{x})}{(1-\gamma)v_1(\hat{x})}\right)^{\frac{1-\gamma\psi}{1-\gamma}}\leq\left(\frac{(1-\gamma)\check{\mathsf{v}}_2(\hat{x})}{(1-\gamma)\check{\mathsf{u}}_1(\hat{x})}\right)^{\frac{1-\gamma\psi}{1-\gamma}},$$

where for the last inequality we used comparisons with sub- and supersolutions constructed in (3.2) (cf. Theorem 3.5). Recall that $b/r\geq 1$, from Theorem 2.1. Moreover $\frac{1-\gamma\psi}{1-\gamma}<0$, hence

$$\left(\frac{b}{r}\right)^{\frac{1-\gamma\psi}{1-\gamma}}\leq 1,\quad\left(\frac{(1-\gamma)\check{\mathsf{v}}_2(\hat{x})}{(1-\gamma)\check{\mathsf{u}}_1(\hat{x})}\right)^{\frac{1-\gamma\psi}{1-\gamma}}=\left(\frac{b}{r}\right)^{\frac{1-\gamma\psi}{1-\gamma}}\left(\frac{r\hat{x}+y_2}{r\hat{x}+y_1}\right)^{1-\gamma\psi}\leq\left(\frac{r\hat{x}+y_2}{r\hat{x}+y_1}\right)^{1-\gamma\psi}.$$

We therefore obtain

$$\frac{r\hat{x}+y_2}{r\hat{x}+y_1}<\left(\frac{r\hat{x}+y_2}{r\hat{x}+y_1}\right)^{1-\gamma\psi}\Rightarrow\left(\frac{r\hat{x}+y_2}{r\hat{x}+y_1}\right)^{\gamma\psi}<1\underbrace{\Rightarrow}_{\gamma\psi>0}\frac{r\hat{x}+y_2}{r\hat{x}+y_1}<1,$$

in contradiction with $y_2>y_1$. Therefore, we conclude $s_1(x)<0$ for all $x>\underline{x}$. Finally, from the state constraint $s_1(\underline{x})\geq 0$ and the continuity of s_1, we obtain $s_1(\underline{x})=0$. □

Corollary 3.18. *If $s_2(\underline{x})>0$, then $Dv_1(\underline{x})>Dv_2(\underline{x})$.*

Proof. From $s_2(\underline{x})>0$, we can differentiate the HJB equation for v_2 at $\underline{x}$. From Theorem 3.17, we can also differentiate the HJB equation satisfied by v_1 at $x>\underline{x}$ and pass to the limit as $x\to\underline{x}$,

$$\begin{aligned}&\left(\rho-r\right)Dv_1(\underline{x})+\lambda_1\left(Dv_1(\underline{x})-Dv_2(\underline{x})\right)\\ =&\lim_{x\to\underline{x}}s_1(x)D^2v_1(x)+\left(1-\frac{1}{\theta}\right)\rho Dv_1(\underline{x})+H_v(\underline{x},y_1,v_1(\underline{x}),Dv_1(\underline{x}))Dv_1(\underline{x}).\end{aligned} \tag{3.28}$$

By subtracting the resulting equations, we obtain

$$\begin{aligned}
&(\rho - r + \lambda_1 + \lambda_2)(Dv_1(\underline{x}) - Dv_2(\underline{x})) \\
&= \lim_{x \to \underline{x}} s_1(x) D^2 v_1(x) - s_2(\underline{x}) D^2 v_2(\underline{x}) + \left(1 - \frac{1}{\theta}\right) \rho (Dv_1(\underline{x}) - Dv_2(\underline{x})) \\
&\quad + H_v(\underline{x}, y_1, v_1(\underline{x}), Dv_1(\underline{x})) Dv_1(\underline{x}) - H_v(\underline{x}, y_2, v_2(\underline{x}), Dv_2(\underline{x})) Dv_2(\underline{x}).
\end{aligned} \tag{3.29}$$

Since $s_1(x) < 0$ we know that $\lim_{x \to \underline{x}} s_1(x) D^2 v_1(x) \geq 0$. On the other hand, $s_2(\underline{x}) > 0$ implies $-s_2(\underline{x}) D^2 v_2(\underline{x}) > 0$ and finally

$$\lim_{x \to \underline{x}} s_1(x) D^2 v_1(x) - s_2(\underline{x}) D^2 v_2(\underline{x}) > 0.$$

By rearranging (3.29) we have

$$\begin{aligned}
&(\rho - r + \lambda_1 + \lambda_2)(Dv_1(\underline{x}) - Dv_2(\underline{x})) - \left(\left(1 - \frac{1}{\theta}\right) \rho + H_v(\underline{x}, y_1, v_1(\underline{x}), Dv_1(\underline{x})) \right) (Dv_1(\underline{x}) - Dv_2(\underline{x})) \\
&= \lim_{x \to \underline{x}} s_1(x) D^2 v_1(x) - s_2(\underline{x}) D^2 v_2(\underline{x}) + \left(H_v(\underline{x}, y_1, v_1(\underline{x}), Dv_1(\underline{x})) - H_v(\underline{x}, y_2, v_2(\underline{x}), Dv_2(\underline{x}))\right) Dv_2(\underline{x}),
\end{aligned} \tag{3.30}$$

$$H_v(\underline{x}, y_1, v_1(\underline{x}), Dv_1(\underline{x})) \leq -\rho \left(1 - \frac{1}{\theta}\right) ((1-\gamma) v_1(\underline{x}))^{\frac{\psi^{-1}-1}{1-\gamma}} (r\underline{x} + y_1)^{1 - \frac{1}{\psi}} \leq -\rho \left(1 - \frac{1}{\theta}\right). \tag{3.31}$$

Next we consider

$$\begin{aligned}
&H_v(\underline{x}, y_1, v_1(\underline{x}), Dv_1(\underline{x})) - H_v(\underline{x}, y_2, v_2(\underline{x}), Dv_2(\underline{x})) \\
&= H_v(\underline{x}, y_1, v_1(\underline{x}), Dv_1(\underline{x})) - H_v(\underline{x}, y_2, v_2(\underline{x}), Dv_1(\underline{x})) + H_v(\underline{x}, y_2, v_2(\underline{x}), Dv_1(\underline{x})) - H_v(\underline{x}, y_2, v_2(\underline{x}), Dv_2(\underline{x})).
\end{aligned}$$

From the mean value theorem, there exists χ_1, $v_1(\underline{x}) < \chi_1 < v_2(\underline{x})$ such that

$$H_v(\underline{x}, y_1, v_1(\underline{x}), Dv_1(\underline{x})) - H_v(\underline{x}, y_2, v_2(\underline{x}), Dv_2(\underline{x})) = H_{vv}(\underline{x}, y_1, \chi_1, Dv_1(\underline{x}))(v_1(\underline{x}) - v_2(\underline{x})).$$

From (2.12), $v_1(\underline{x}) < v_2(\underline{x})$ and $Dv_2(\underline{x}) \geq 0$ we have

$$H_{vv}(\underline{x}, y_1, \chi_1, Dv_1(\underline{x}))(v_1(\underline{x}) - v_2(\underline{x})) Dv_2(\underline{x}) \geq 0.$$

For some $\xi \in (0,1)$ and $\chi_2 = \xi Dv_1(\underline{x}) + (1-\xi) Dv_2(\underline{x})$,

$$H_v(\underline{x}, y_2, v_2(\underline{x}), Dv_1(\underline{x})) - H_v(\underline{x}, y_2, v_2(\underline{x}), Dv_2(\underline{x})) = H_{vp}(\underline{x}, y_2, v_2(\underline{x}), \chi_2)(Dv_1(\underline{x}) - Dv_2(\underline{x})).$$

From (2.11), $H_{vp}(\underline{x}, y_2, v_2(\underline{x}), \chi_2) Dv_2(\underline{x}) \leq 0$. Combining (3.30) and the observation above yields $Dv_1(\underline{x}) > Dv_2(\underline{x})$. ☐

A consequence of the above results is that the value of the unproductive agents is singular at the borrowing limit.

Corollary 3.19. *We have*

$$\lim_{x \to \underline{x}} D^2 v_1(x) = -\infty. \tag{3.32}$$

Proof. We deduce from (3.31) that

$$\left(1-\frac{1}{\theta}\right)\rho Dv_1(\underline{x})+H_v(\underline{x},y_1,v_1(\underline{x}),Dv_1(\underline{x}))Dv_1(\underline{x})\leq 0.$$

With this inequality, $\rho \geq r$, Theorem 3.18 and (3.28) lead to

$$0<(\rho-r)Dv_1(\underline{x})+\lambda_1(Dv_1(\underline{x})-Dv_2(\underline{x}))\leq \lim_{x\to\underline{x}} s_1(x)D^2v_1(x).$$

This implies the desired result. □

The following proposition provides a sufficient condition under which the savings of productive agents are negative for all $x>\underline{x}$. In this regime, productive agents decumulate capital regardless of their wealth.

Proposition 3.20. *Suppose $r>0$ and*

$$\left(\frac{\rho}{\theta}-r\right)(r\underline{x}+y_2)^{-1/\psi}+\lambda_2\left(\underbrace{(r\underline{x}+y_2)^{-1/\psi}-(r\underline{x}+y_1)^{-1/\psi}}_{<0}\right)\geq 0. \tag{3.33}$$

Then $s_2(x)<0$ for all $x>\underline{x}$ and $s_2(\underline{x})=0$.

Proof. We argue by contradiction and suppose that there exists $\hat{x}>\underline{x}$ such that $s_2(\hat{x})\geq 0$. Then

$$c_2(\hat{x})\leq r\hat{x}+y_2 \quad\text{and}\quad Dv_2(\hat{x})\geq \rho(r\hat{x}+y_2)^{-1/\psi}((1-\gamma)v_2(\hat{x}))^{\frac{1-\gamma\psi}{\psi(1-\gamma)}}.$$

From Theorem 3.17, $s_1(\hat{x})<0$, hence

$$c_1(\hat{x})> r\hat{x}+y_1 \quad\text{and}\quad Dv_1(\hat{x})< \rho(r\hat{x}+y_1)^{-1/\psi}((1-\gamma)v_1(\hat{x}))^{\frac{1-\gamma\psi}{\psi(1-\gamma)}}.$$

With Theorem 3.15, if $s_2(\hat{x})>0$ we differentiate the HJB equation for v_2 at $\hat{x}$ to obtain

$$\left(\frac{\rho}{\theta}-r\right)Dv_2(\hat{x})+\lambda_2\left(Dv_2(\hat{x})-Dv_1(\hat{x})\right)=s_2(\hat{x})D^2v_2(\hat{x})+H_v(\hat{x},y_2,v_2(\hat{x}),Dv_2(\hat{x}))Dv_2(\hat{x})\leq 0. \tag{3.34}$$

If $s_2(\hat{x})=0$ we obtain (3.34) by differentiating in a neighborhood of $\hat{x}$ and pass to the limit.

Let us consider the case $\frac{\rho}{\theta}-r+\lambda_2\leq 0$. In this case,

$$\left(\frac{\rho}{\theta}-r\right)(r\hat{x}+y_2)^{-1/\psi}+\lambda_2\left(\underbrace{(r\hat{x}+y_2)^{-1/\psi}-(r\hat{x}+y_1)^{-1/\psi}}_{<0}\right)< 0, \tag{3.35}$$

hence

$$\frac{\frac{\rho}{\theta}-r+\lambda_2}{\lambda_2}<\left(\frac{r\hat{x}+y_2}{r\hat{x}+y_1}\right)^{1/\psi}<\left(\frac{r\underline{x}+y_2}{r\underline{x}+y_1}\right)^{1/\psi}, \tag{3.36}$$

where for the last inequality we used $y_2>y_1$, $r\hat{x}>r\underline{x}$, $r\underline{x}+y_j>0$ and $1/\psi>0$. It is easy to see that (3.36) contradicts (3.33).

Now let us consider the case when $\frac{\rho}{\theta} - r + \lambda_2 > 0$. From (3.34), we obtain that

$$\left(\frac{\rho}{\theta} - r + \lambda_2\right)(r\hat{x} + y_2)^{-1/\psi}\left(\frac{v_2(\hat{x})}{v_1(\hat{x})}\right)^{\frac{1-\gamma\psi}{\psi(1-\gamma)}} - \lambda_2(r\hat{x} + y_1)^{-1/\psi} < 0.$$

Since $v_1(\hat{x}) < v_2(\hat{x}) < 0$ and $\frac{1-\gamma\psi}{\psi(1-\gamma)} < 0$, we observe that $\frac{v_2(\hat{x})}{v_1(\hat{x})} < 1$, thus $\left(\frac{v_2(\hat{x})}{v_1(\hat{x})}\right)^{\frac{1-\gamma\psi}{\psi(1-\gamma)}} > 1$. We recover (3.35) and then (3.36), in contradiction with (3.33). We have therefore proved that $s_2(x) < 0$ for all $x > \underline{x}$ and conclude $s_2(\underline{x}) = 0$, from the state constraint condition and the continuity of s_2. □

By contrast with Theorem 3.20, the following proposition contains a sufficient condition for the savings of the productive agents at the borrowing limit to be positive.

Proposition 3.21. *Suppose* $0 \le r \le \rho$ *and*

$$(\rho - r)(r\underline{x} + y_2)^{-1/\psi} + \lambda_2\left(\underbrace{(r\underline{x} + y_2)^{-1/\psi} - (r\underline{x} + y_1)^{-1/\psi}}_{<0}\right) < 0. \tag{3.37}$$

Then $s_2(\underline{x}) > 0$.

Proof. We argue by contradiction. Suppose $s_2(\underline{x}) = 0$. Since $c_2(\underline{x}) = r\underline{x} + y_2$ and using the comparison with supersolution found in Theorem 3.5,

$$H_v(\underline{x}, y_2, v_2(\underline{x}), Dv_2(\underline{x})) \ge -\left(1 - \frac{1}{\theta}\right)\rho\underbrace{\left(\frac{\rho}{r}\right)^{-\theta}}_{\le 1} \ge -\left(1 - \frac{1}{\theta}\right)\rho.$$

Moreover $s_1(\underline{x}) = 0$ and $s_2(\underline{x}) = 0$ yield

$$Dv_1(\underline{x}) = \rho(r\underline{x} + y_1)^{-1/\psi}((1-\gamma)v_1(\underline{x}))^{\frac{1-\gamma\psi}{\psi(1-\gamma)}} \text{ and } Dv_2(\underline{x}) = \rho(r\underline{x} + y_2)^{-1/\psi}((1-\gamma)v_2(\underline{x}))^{\frac{1-\gamma\psi}{\psi(1-\gamma)}}. \tag{3.38}$$

From the continuity of s_2, we may define

$$\delta^+ = \max\{\delta \ge 0 : \quad s_2(x) = 0 \ \forall x \in [\underline{x}, \underline{x} + \delta]\},$$

and consider different cases.

Case (I): If $\delta^+ > 0$, then

$$Dv_2(x) = \rho(rx + y_2)^{-1/\psi}((1-\gamma)v_2(x))^{\frac{1-\gamma\psi}{\psi(1-\gamma)}}$$

for $x \in (\underline{x}, \underline{x} + \delta^+]$ and

$$\begin{aligned} D^2 v_2(x) = & -\frac{r\rho}{\psi}(rx + y_2)^{-\frac{1}{\psi}-1}((1-\gamma)v_2(x))^{\frac{1-\gamma\psi}{\psi(1-\gamma)}} \\ & + \frac{\rho(1-\gamma\psi)}{\psi}(rx + y_2)^{-1/\psi}((1-\gamma)v_2(x))^{\frac{1-\gamma\psi}{\psi(1-\gamma)}-1} \\ \ge & -\frac{r\rho}{\psi}(\rho x + y_2)^{-\gamma} \ge -\frac{r\rho}{\psi}(\rho\underline{x} + y_2)^{-\gamma}, \end{aligned}$$

for $x \in (\underline{x}, \underline{x}+\delta^+)$. Hence, $\lim_{x \to \underline{x}} s_2(x) D^2 v_2(x) = 0$ and

$$\begin{aligned}&(\rho - r)\, Dv_2(\underline{x}) + \lambda_2\,(Dv_2(\underline{x}) - Dv_1(\underline{x})) \\ = &\lim_{x \to \underline{x}} s_2(x) D^2 v_2(x) + \left(1 - \frac{1}{\theta}\right) \rho Dv_2(\hat{x}) + H_v(\underline{x}, y_2, v_2(\underline{x}), Dv_2(\underline{x})) Dv_2(\underline{x}),\end{aligned}$$

hence

$$(\rho - r)\, Dv_2(\underline{x}) + \lambda_2\,(Dv_2(\underline{x}) - Dv_1(\underline{x})) \geq 0. \tag{3.39}$$

From (3.39), (3.38) and $v_1(\underline{x}) < v_2(\underline{x})$, we then find the desired contradiction with (3.37).

Case (II): Now we consider the case $\delta^+ = 0$.

Case $(II1)$: We first consider the case that $\delta^+ = 0$ and there exists a sufficiently small $\varepsilon > 0$ such that either $s_2(x) > 0$ or $s_2(x) < 0$ in $(\underline{x}, \underline{x}+\varepsilon)$.

(1): Suppose $s_2(x) < 0$ in $(\underline{x}, \underline{x}+\varepsilon)$ then $\liminf_{x \to \underline{x}} s_2(x) D^2 v_2(x) \geq 0$, therefore we have (3.39) and hence a contradiction with (3.37).

(2): Suppose $s_2(x) > 0$ in $(\underline{x}, \underline{x}+\varepsilon)$, we claim that it is impossible that $\lim_{x \to \underline{x}} D^2 v_2(x) = -\infty$. If $\lim_{x \to \underline{x}} D^2 v_2(x) = -\infty$, there exists $\hat{x} \in (\underline{x}, \underline{x}+\varepsilon)$ such that

$$\begin{aligned}Ds_2(x) = r &+ \psi \rho^{\psi} (Dv_2(x))^{-\psi-1} ((1-\gamma) v_2(x))^{\frac{1-\gamma\psi}{1-\gamma}} D^2 v_2(x) \\ &- (1-\gamma\psi) \rho^{\psi} (Dv_2(x))^{1-\psi} ((1-\gamma) v_2(x))^{\frac{1-\gamma\psi}{1-\gamma}-1} < 0, \quad \forall x \in (\underline{x}, \hat{x}).\end{aligned} \tag{3.40}$$

Therefore, $s_2(\hat{x}) < 0$, in contradiction with $s_2(x) > 0$ for $x \in (\underline{x}, \underline{x}+\varepsilon)$. Since $\lim_{x \to \underline{x}} D^2 v_2(x) = -\infty$ is not possible, we know there exists a sequence $x_n \to \underline{x}$ such that

$$\lim_{x_n \to \underline{x}} s_2(x_n) D^2 v_2(x_n) = 0$$

and we again obtain (3.39), in contradiction with (3.37).

Case $(II2)$: We consider the case that $\delta^+ = 0$ and there exists $x_n \to \underline{x}$ such that $s_2(x_n) = 0$. Again, we claim it is impossible that $\lim_{x_n \to \underline{x}} D^2 v_2(x_n) = -\infty$. Otherwise, for sufficiently small x_n we can show, similarly to the calculation in (3.40), that $Ds_2(x) < 0$ for all $x \in (\underline{x}, x_n)$, hence $s_2(x_n) < 0$. Having ruled out the possibility that $\lim_{x_n \to \underline{x}} D^2 v_2(x_n) = -\infty$, we can extract a bounded subsequence of $D^2 v_2(x_n)$ and obtain (3.39), in contradiction with (3.37). ☐

The next result deals with the behavior of s_2 as $x \to +\infty$.

Proposition 3.22. *Suppose $\rho > r$ and $s_2(\underline{x}) > 0$. There exists $\bar{x} > \underline{x}$ such that $s_2(\bar{x}) = 0$ and $s_2(x) < 0$ for all $x \geq \bar{x}$.*

Proof. Let us assume that for all $\hat{x}$ sufficiently large, $s_2(\hat{x}) > 0$ and look for a contradiction. Differentiating the HJB equation for v_2 at $\hat{x}$ leads to (3.20). From $s_2(\hat{x}) > 0$, we obtain $c_2(\hat{x}) < r\hat{x} + y_2$. The comparison with subsolution found in Theorem 3.5 yields

$$v_2(\hat{x}) \geq \frac{(r\hat{x} + y_1)^{1-\gamma}}{1-\gamma}.$$

From the monotonicity of H_v ((2.11) and (2.12)), we obtain

$$H_v(\hat{x}, y_2, v_2(\hat{x}), Dv_2(\hat{x})) < -\rho\left(1-\frac{1}{\theta}\right)\left(\frac{r\hat{x}+y_2}{r\hat{x}+y_1}\right)^{1-\psi^{-1}}.$$

We deduce from (3.20) that

$$\rho - r + \lambda_2 < \rho\left(1-\frac{1}{\theta}\right)\left(1-\left(\frac{r\hat{x}+y_2}{r\hat{x}+y_1}\right)^{1-\psi^{-1}}\right) + \frac{\lambda_2 Dv_1(\hat{x})}{Dv_2(\hat{x})}.$$

From Theorem 3.17, $c_1(\hat{x}) > r\hat{x} + y_1$, hence

$$\frac{Dv_1(\hat{x})}{Dv_2(\hat{x})} = \left(\frac{c_2(\hat{x})}{c_1(\hat{x})}\right)^{1/\psi}\left(\frac{v_1(\hat{x})}{v_2(\hat{x})}\right)^{\frac{1-\gamma\psi}{\psi(1-\gamma)}} < \left(\frac{r\hat{x}+y_2}{r\hat{x}+y_1}\right)^{\psi^{-1}},$$

therefore

$$\rho - r < \rho\left(1-\frac{1}{\theta}\right)\left(1-\left(\frac{r\hat{x}+y_2}{r\hat{x}+y_1}\right)^{1-\psi^{-1}}\right) + \lambda_2\left(\left(\frac{r\hat{x}+y_2}{r\hat{x}+y_1}\right)^{\psi^{-1}}-1\right).$$

Passing to the limit we obtain

$$\lim_{\hat{x}\to+\infty}\rho\left(1-\frac{1}{\theta}\right)\left(1-\left(\frac{r\hat{x}+y_2}{r\hat{x}+y_1}\right)^{1-\psi^{-1}}\right) + \lambda_2\left(\left(\frac{r\hat{x}+y_2}{r\hat{x}+y_1}\right)^{\psi^{-1}}-1\right) = 0,$$

a contradiction with $\rho > r$. The existence of $\bar{x}$ then follows from the continuity of s_2. ☐

4 Analysis of the Fokker-Planck-Kolmogorov equation

Let us state the weak formulation of the system of Fokker-Planck-Kolmogorov equations:

$$-\frac{\partial}{\partial x}\left[(rx + y_j - c_j(x))m_j(x)\right] + \lambda_{\bar{j}} m_{\bar{j}}(x) - \lambda_j m_j(x) = 0. \tag{4.1}$$

Definition 4.1. The measure m with $m = \sum_{j\in\{1,2\}} m_j \otimes \delta_{y_j}(y)$ is a weak solution to (4.1) if for all test functions $(\phi_1, \phi_2) \in \left(C_c^1([\underline{x}, +\infty))\right)^2$ and $j \in \{1,2\}$,

$$\int_{x\geq\underline{x}} \lambda_j \phi_j(x) dm_j - \int_{x\geq\underline{x}} \lambda_{\bar{j}} \phi_j(x) dm_{\bar{j}} = \int_{x\geq\underline{x}} s_j(x) D\phi_j(x) dm_j. \tag{4.2}$$

In the stationary regime, the singularities in the wealth distribution may appear only at $\underline{x}$ or $\hat{x} > \underline{x}$ such that $s_1(\hat{x}) = 0$ or $s_2(\hat{x}) = 0$. It has been proven in Theorem 3.17 that s_1 does not vanish in $(0, +\infty)$. On the other hand, the asymptotics analysis at the points where s_2 vanishes show that, with the optimal strategy, it takes an infinite time to reach these points. Therefore, the wealth distribution of the population labeled j has a density g_j in $(\underline{x}, +\infty)$ and may exhibit a Dirac mass at $\underline{x}$, of the form $\mu_j\delta_{\underline{x}}$. We can then write the probability measure as

$$dm_j = g_j dx + \mu_j \delta_{\underline{x}}. \tag{4.3}$$

We denote by $G_j(x)$ the cumulative distribution function:

$$G_j(x) := \mu_j + \int_{\underline{x}}^{x} g_j(x)dx. \tag{4.4}$$

We may rewrite (4.2) as follows

$$\int_{x>\underline{x}} \left(\lambda_j g_j(x) - \lambda_{\bar{j}} g_{\bar{j}}(x)\right) \phi_j(x)dx + (\lambda_j\mu_j - \lambda_{\bar{j}}\mu_{\bar{j}})\phi_j(\underline{x}) = \int_{x>\underline{x}} s_j(x)g_j(x)D\phi_j(x)dx + \mu_j s_j(\underline{x})D\phi_j(\underline{x}).$$

Lemma 4.2. *Given any r such that $0 \le r < \rho$, let $m = \sum_{j\in\{1,2\}} m_j \otimes \delta_{y_j}(y)$ be a weak solution to the FPK equation. Then we have*

$$\lim_{x\to+\infty} G_j(x) = \int_{\underline{x}}^{+\infty} dm_j = \mu_j + \int_{\underline{x}}^{+\infty} g_j(x)dx = \frac{\lambda_{\bar{j}}}{\lambda_j + \lambda_{\bar{j}}}.$$

Proof. By taking the test functions $\phi_j = 1$ in (4.2) we have

$$\lambda_1 \int_{\underline{x}}^{+\infty} dm_1 = \lambda_2 \int_{\underline{x}}^{+\infty} dm_2, \tag{4.5}$$

we then obtain (4.5) from $\int_{\underline{x}}^{+\infty} dm_1 + \int_{\underline{x}}^{+\infty} dm_2 = 1$. $\square$

Let $\bar{x}$ be defined as in Theorem 3.22. We first observe $g_1(x) = g_2(x) = 0$ for all x such that $x > \bar{x}$, because the optimal policies consist of decumulating wealth when $x > \bar{x}$, so the stationary distribution must vanish in this region (cf. [31]).

Lemma 4.3. *Assume r satisfies $0 \le r < \rho$ and $s_2(\underline{x}) > 0$. Then we have*

$$s_1(x)g_1(x) + s_2(x)g_2(x) = 0 \quad \forall x > \underline{x}. \tag{4.6}$$

Proof. From the FPK equation we can derive for all $x > \underline{x}$,

$$\frac{d}{dx}\left(s_1(x)g_1(x) + s_2(x)g_2(x)\right) = 0.$$

Moreover for $x > \bar{x}$, $s_1(x)g_1(x) + s_2(x)g_2(x) = 0$, therefore we obtain (4.6). $\square$

If $s_2(\underline{x}) > 0$, from Theorem 3.22 and the continuity of s_2, we know that there exists $\widehat{x}$

$$\widehat{x} = \min_x\{x : \ s_2(x) = 0\}. \tag{4.7}$$

It is clear that $\widehat{x} \le \bar{x}$ and $s_2(\widehat{x}) = 0$. Moreover, $s_2(x) > 0$ for all $x < \widehat{x}$. We recall that $\bar{x}$ in Theorem 3.22 is the last point where s_2 vanishes, i.e. $s_2(x) < 0$ for all $x > \bar{x}$.

Proposition 4.4. *Assume $s_2(\underline{x}) > 0$ and let $\widehat{x}$ be defined by (4.7). Then, there exists κ_2 such that the densities are given by*

$$\text{For all } x \in (\underline{x}, \widehat{x}), \quad g_2(x) = \frac{\kappa_2}{s_2(x)} \exp\left(\int_{\underline{x}}^{x} \left(-\frac{\lambda_1}{s_1(z)} - \frac{\lambda_2}{s_2(z)}\right) dz\right), \tag{4.8}$$

$$\text{For all } x \in (\underline{x}, \widehat{x}), \quad g_1(x) = -\frac{\kappa_2}{s_1(x)} \exp\left(\int_{\underline{x}}^{x} \left(-\frac{\lambda_1}{s_1(z)} - \frac{\lambda_2}{s_2(z)}\right) dz\right). \tag{4.9}$$

Proof. For all $x \in (\underline{x}, \widehat{x})$, $s_2(x) > 0$ and $s_1(x) < 0$, we can write

$$\frac{d}{dx}\left(s_2(x)g_2(x)\right) = \left(-\frac{\lambda_1}{s_1(x)} - \frac{\lambda_2}{s_2(x)}\right)\left(s_2(x)g_2(x)\right). \tag{4.10}$$

From $s_2(\underline{x}) > 0$ and $s_1(x) = O(\sqrt{x - \underline{x}})$ near $\underline{x}$ (see Appendix C), we infer $-\frac{\lambda_1}{s_1(x)} - \frac{\lambda_2}{s_2(x)}$ is integrable in a neighborhood of $\underline{x}$, which allows us to integrate (4.9) in $[\underline{x}, x)$ for any $x \in [\underline{x}, \widehat{x})$. We obtain that there exists $\kappa_2 > 0$ such that for all $x \in [\underline{x}, \widehat{x})$,

$$g_2(x) = \frac{\kappa_2}{s_2(x)} \exp\left(\int_{\underline{x}}^{x} -\frac{\lambda_1}{s_1(z)} - \frac{\lambda_2}{s_2(z)} dz\right).$$

It is clear that $\kappa_2 = g_2(\underline{x})s_2(\underline{x})$. We then deduce (4.9) in $(\underline{x}, \widehat{x})$ from (4.6). ☐

Proposition 4.5. *Assume $s_2(\underline{x}) > 0$ and let $\widehat{x}$ be defined by (4.7). Then, $g_j(x) = 0$ for all $x > \widehat{x}$.*

Proof. The densities satisfy $g_j(x) = 0$ on the set $\{x : \widehat{x} < x < \bar{x},\ s_2(x) < 0\}$. This follows from Theorem 3.17 and Theorem 4.3.

Next, we show the densities $g_j(x) = 0$ on the set $\{x : \widehat{x} < x < \bar{x},\ s_2(x) > 0\}$. This is an open set, i.e. a countable or finite union of disjoint intervals $I_k = (a_k, b_k)$. We know that, if $a_k \neq \bar{x}$ then

$$s_2(a_k) = s_2(b_k) = 0, \quad Ds_2(a_k) \geq 0. \tag{4.11}$$

If $s_2(a_k) = Ds_2(a_k) = 0$, then $s_2(x) = o(x - a_k)$. Therefore, (4.11) implies $s_2(x) = O(x - a_k)$ as $x \to a_{k,+}$. We now choose $\xi \in I_k$ and denote $C_k = s_2(\xi)g_2(\xi)$. If $g_2(\xi) > 0$, then by using (4.10), we obtain

$$g_2(x) = \frac{C_k}{s_2(x)} \exp\left(\int_{\xi}^{x}\left(-\frac{\lambda_1}{s_1(z)} - \frac{\lambda_2}{s_2(z)}\right)dz\right),\ g_1(x) = -\frac{C_k}{s_1(x)} \exp\left(\int_{\xi}^{x}\left(-\frac{\lambda_1}{s_1(z)} - \frac{\lambda_2}{s_2(z)}\right)dz\right). \tag{4.12}$$

We note that $\xi > a_k$, $s_1(x)$ is bounded away from 0 if $x > \widehat{x}$. We thus infer that $\exp\left(\int_{\xi}^{x}\left(-\frac{\lambda_1}{s_1(z)}\right)dz\right)$ is bounded and $\exp\left(\int_{\xi}^{x}\left(-\frac{\lambda_2}{s_2(z)}\right)dz\right) \geq 0$ for all x in a right neighborhood of a_k. We then deduce from $s_2(x) = O(x - a_k)$ that the density g_2 proposed in (4.12) is not integrable near $a_{k,+}$. Therefore $C_k = 0$ and $g_2(x) = 0$ for all $x \in I_k$. Finally, (4.6) and $s_1(x) < 0$ give $g_1(x) = 0$ for all $x \in I_k$.

Finally, we consider $\check{x}$, such that $s_2(x) = 0$ in a neighborhood of $\check{x}$. We deduce from (4.6), $s_2(\check{x}) = 0$ and $s_1(\check{x}) < 0$ that $g_1(\check{x}) = 0$. From (1.5), we obtain $g_2(\check{x}) = 0$. ☐

Corollary 4.6. *Assume $s_2(\underline{x}) > 0$. We have*

$$\kappa_2 = \lambda_1 \mu_1.$$

Proof. Note that $\mu_2 = 0$ because $s_2(\underline{x}) > 0$. From (4.5) and $\mu_2 = 0$, we deduce

$$-\lambda_1\mu_1 = \lambda_1 \int_{\underline{x}}^{\widehat{x}} g_1(x)dx + \lambda_2 \int_{\underline{x}}^{\widehat{x}} g_2(x)dx.$$

By integrating the FPK equation for g_2, we then obtain

$$-\lambda_1\mu_1 = \int_{\underline{x}}^{\widehat{x}+1} \frac{d}{dx}\left(s_2(z)g_2(z)\right)dz = s_2(\widehat{x}+1)g_2(\widehat{x}+1) - \lim_{x \to \underline{x}} s_2(x)g_2(x).$$

From Theorem 4.5, $s_2(\widehat{x}+1)g_2(\widehat{x}+1) = 0$. The result follows from (4.8). ☐

Proposition 4.7. *There exists $\hat{r} > 0$, for all r such that $\hat{r} \le r < \rho$, the aggregate wealth $\mathcal{K}[r]$ is positive and depends continuously on r.*

Proof. Let $r^{(\iota)} \to r$ with $r < \rho$, we have $s_1^{(\iota)}(x) < 0$ and $s_2^{(\iota)}(\underline{x}) > 0$. From Theorem 3.13 we know that $s_j^{(\iota)}$ converges to s_j locally uniformly. For any $x > \widehat{x}$ such that $s_2(x) < 0$, we can obtain $s_j^{(\iota)}(x) < 0$ with ι sufficiently large. For any $M > \widehat{x}$, for ι sufficiently large, the measure $m_j^{(\iota)}$ is supported in $[\underline{x}, M]$. Hence we can extract a subsequence $(m_1^{(\iota)}, m_2^{(\iota)})$ which converges weakly in the sense of measure and the limit $(\tilde{m}_1, \tilde{m}_2)$ is supported in $[\underline{x}, \widehat{x}]$. By passing to the limit in the weak form of FPK equation (4.2), we obtain that for all test functions $(\phi_1, \phi_2) \in \left(C_c^1([\underline{x}, M))\right)^2$,

$$\int_{x \ge \underline{x}} \lambda_j \phi_j(x) d\tilde{m}_j - \int_{x \ge \underline{x}} \lambda_{\bar{j}} \phi_j(x) d\tilde{m}_{\bar{j}} = \int_{x \ge \underline{x}} s_j(x) D\phi_j(x) d\tilde{m}_j. \tag{4.13}$$

The solution of (4.13) is unique, namely given (m_1, m_2) defined by (4.3). Therefore, the whole sequence $(m_1^{(\iota)}, m_2^{(\iota)})$ converges weakly to (m_1, m_2). Recalling (1.8), we see that

$$\mathcal{K}[r^{(\iota)}] = \sum_{j \in \{1,2\}} \int_{\underline{x}}^{M} x dm_j^{(\iota)},$$

and conclude that $\mathcal{K}[r^{(\iota)}]$ converges to $\mathcal{K}[r]$. ☐

With similar arguments as above, we can study the situation when $s_2(\underline{x}) = 0$:

Proposition 4.8. *If $s_2(\underline{x}) = 0$, then $m_1 = \frac{\lambda_2}{\lambda_1+\lambda_2}\delta_{\underline{x}}$ and $m_2 = \frac{\lambda_1}{\lambda_1+\lambda_2}\delta_{\underline{x}}$.*

5 Existence of solution to the Mean Field Game system

5.1 Nonexistence of invariant measures when $r = \rho$

In Section 4 we have shown that the aggregate wealth depends continuously on the interest rate r. We now show that the aggregate wealth blows up as $r \to \rho$. For this purpose, we consider the limiting case $r = \rho$. For b defined in Table 1 and $b = \rho$, the sub and supersolutions given in Theorem 3.5 become:

$$(\check{\mathsf{u}}_1, \check{\mathsf{u}}_1) = \left(\frac{(\rho x + y_1)^{1-\gamma}}{1-\gamma}, \frac{(\rho x + y_1)^{1-\gamma}}{1-\gamma}\right), \quad (\check{\mathsf{v}}_2, \check{\mathsf{v}}_2) = \left(\frac{(\rho x + y_2)^{1-\gamma}}{1-\gamma}, \frac{(\rho x + y_2)^{1-\gamma}}{1-\gamma}\right). \tag{5.1}$$

Proposition 5.1. *If $r = \rho$, then $s_2(x) > 0$ for all $x \ge \underline{x}$.*

Proof. It is clear from Theorem 3.21 that $s_2(\underline{x}) > 0$ if $r = \rho$. Next we argue by contradiction and suppose $s_2(\hat{x}) \le 0$ for some $\hat{x} > \underline{x}$.

Step 1. Suppose $s_2(\hat{x}) < 0$. From Theorem 3.15, we can differentiate (3.19) and (3.20) at $\hat{x}$:

$$\lambda_1 \left(Dv_1(\hat{x}) - Dv_2(\hat{x})\right) = s_1(\hat{x}) D^2 v_1(\hat{x}) + \left(1 - \frac{1}{\theta}\right) \rho Dv_1(\hat{x}) + H_v(\hat{x}, y_1, v_1(\hat{x}), Dv_1(\hat{x})) Dv_1(\hat{x}), \tag{5.2}$$

$$\lambda_2 \left(Dv_2(\hat{x}) - Dv_1(\hat{x})\right) = s_2(\hat{x}) D^2 v_2(\hat{x}) + \left(1 - \frac{1}{\theta}\right) \rho Dv_2(\hat{x}) + H_v(\hat{x}, y_2, v_2(\hat{x}), Dv_2(\hat{x})) Dv_2(\hat{x}). \tag{5.3}$$

From (2.11), (2.12), (5.1) and $s_2(\hat{x}) < 0$ we can infer

$$H_v(\hat{x}, y_2, v_2(\hat{x}), Dv_2(\hat{x})) \geq -\left(1 - \frac{1}{\theta}\right)\rho,$$

hence

$$\left(1 - \frac{1}{\theta}\right)\rho Dv_2(\hat{x}) + H_v(\hat{x}, y_2, v_2(\hat{x}), Dv_2(\hat{x}))Dv_2(\hat{x}) \geq 0. \tag{5.4}$$

Inequality (5.4), with $s_2(\hat{x})D^2v_2(\hat{x}) \geq 0$ and (5.3) lead to $Dv_2(\hat{x}) \geq Dv_1(\hat{x})$. This and $v_2(\hat{x}) > v_1(\hat{x})$ yield

$$H_v(\hat{x}, y_1, v_1(\hat{x}), Dv_1(\hat{x})) > H_v(\hat{x}, y_2, v_2(\hat{x}), Dv_2(\hat{x})) \geq -\left(1 - \frac{1}{\theta}\right)\rho,$$

hence

$$\left(1 - \frac{1}{\theta}\right)\rho Dv_1(\hat{x}) + H_v(\hat{x}, y_1, v_1(\hat{x}), Dv_1(\hat{x}))Dv_1(\hat{x}) > 0.$$

From (5.2), we obtain $s_1(\hat{x})D^2v_1(\hat{x}) < 0$. Since $D^2v_1(\hat{x}) < 0$, this is in contradiction with $s_1(\hat{x}) < 0$ (obtained in Theorem 3.17).

Step 2. Suppose $s_2(\hat{x}) = 0$, we can differentiate (3.20) in a neighborhood of $\hat{x}$. Moreover, from Theorem 3.15 we have $\lim_{x\to\hat{x}} s_2(x)D^2v_2(x) = 0$. We proceed similarly as in *Step 1* and omit the details. □

The following proposition describes the asymptotics of the savings in the limit $x \to +\infty$. The proof is in Appendix B.

Proposition 5.2. *Assume that $r = \rho$. As $x \to +\infty$,*

$$\begin{aligned} & s_j(x) \\ = & \frac{\lambda_j(y_j - y_{\bar{j}})}{\rho + \lambda_j + \lambda_{\bar{j}}} + o(1) \\ = & \frac{\lambda_j(y_j - y_{\bar{j}})}{\rho + \lambda_j + \lambda_{\bar{j}}} + \frac{\gamma(1+\psi)\lambda_1(y_{\bar{j}} - y_j)^2}{2(\rho + \lambda_j + \lambda_{\bar{j}})}\left[1 - \frac{\lambda_j\lambda_{\bar{j}} + \lambda_{\bar{j}}^2 + \rho\lambda_j}{(\rho + \lambda_j + \lambda_{\bar{j}})^2} - \frac{\lambda_j}{(\rho + \lambda_j + \lambda_{\bar{j}})}\right](\rho x + y_j)^{-1} \\ & + o((\rho x + y_1)^{-1}). \end{aligned} \tag{5.5}$$

Theorem 5.2 allows us to show that when $r = \rho$, there is no stationary measure. The following result is proved in Appendix B.

Proposition 5.3. *Assume that $r = \rho$. Let v_j solve the HJB equations (1.4). The FPK equation (4.1) does not have a solution. In other words, there is no stationary probability measure when $r = \rho$.*

In the next result, we show $\mathcal{K}(r)$ becomes unbounded as $r \to \rho$.

Corollary 5.4. *For any constant $C_{\mathcal{K}} > 0$, there exists $\bar{r} \in [0, \rho)$ such that the aggregate wealth*

$$\mathcal{K}(\bar{r}) > C_{\mathcal{K}}. \tag{5.6}$$

Proof. Suppose (5.6) does not hold and we consider a sequence $r^{(\iota)} \to \rho$, $r^{(\iota)} < \rho$, such that

$$\mathcal{K}(r^{(\iota)}) \leq C_{\mathcal{K}}.$$

It is easy to obtain for all ι,

$$\int_{x \geq 0} |x| dm_1^{(\iota)} + \int_{x \geq 0} |x| dm_2^{(\iota)} \leq C_{\mathcal{K}} + |\underline{x}|, \quad \int_{x \geq \underline{x}} |x| dm_1^{(\iota)} + \int_{x \geq \underline{x}} |x| dm_2^{(\iota)} \leq C_{\mathcal{K}} + 2|\underline{x}|.$$

It follows that the sequence of probability measures $m^{(\iota)}$ is tight (cf. [31, Proposition D.5.2 and Lemma D.5.3]). From Prokhorovs theorem, we can extract a weakly convergent subsequence of $m^{(\iota)}$. By passing to the limit in the weak form of FPK equation (4.2), we obtain a solution to (4.2) with $r = \rho$ and a density defined by (4.3), in contradiction with Theorem 5.3. ☐

5.2 Existence of solutions to the MFG system

We now consider the existence of solutions to the Aiyagari model.

Theorem 5.5. *Suppose*

$$\frac{\rho}{\theta \lambda_2} > \left(\frac{y_2}{y_1} \right)^{1/\psi} - 1. \tag{5.7}$$

There exists a solution to the Aiyagari model (1.7)-(1.11) *with the equilibrium interest rate* r^* *such that* $0 < r^* < \rho$.

Proof. We have shown, with Theorem 3.8, the existence and uniqueness of the solution to the HJB equation and optimal saving policy $s_j^{(r)}$ for each r. We recall the notations for the aggregate asset $K[m]$ and $\mathcal{K}(r)$ given by (1.6) and (1.8). We rewrite the equilibrium condition for the Aiyagari model (1.11) as

$$r^* = \mathcal{B}(r^*), \quad \text{s.t. } \mathcal{B}(r) = A\alpha \left(\frac{\mathcal{K}(r)}{N} \right)^{\alpha - 1} - \delta.$$

It follows directly from Theorem 4.7 that $\mathcal{B}(r)$ depends continuously on r. Let us take

$$C_{\mathcal{K}} = \left(\frac{\delta}{A\alpha} \right)^{\frac{1}{\alpha - 1}} N.$$

From Theorem 5.4, there exists $\bar{r} > 0$ such that $\mathcal{K}(\bar{r}) > C_{\mathcal{K}}$, hence $\bar{r} - \mathcal{B}(\bar{r}) > 0$.

From condition (5.7), there exists $\hat{r} > 0$ sufficiently small such that

$$\frac{\rho - \hat{r}\theta}{\theta \lambda_2} > \left(\frac{\hat{r}\underline{x} + y_2}{\hat{r}\underline{x} + y_1} \right)^{1/\psi} - 1. \tag{5.8}$$

We use $s_j^{(\hat{r})}$ to denote the saving policies corresponding to $r = \hat{r}$. With Theorem 3.17 and Theorem 3.20, we have $s_j^{(\hat{r})}(\underline{x}) = 0$, $s_j^{(\hat{r})}(x) < 0$ for all $x > \underline{x}$. This gives $\mathcal{K}(\hat{r}) = \underline{x} \leq 0$. From the continuity of $\mathcal{K}(r)$, given by Theorem 4.7, there exists r_0, $\hat{r} \leq r_0 < \rho$ such that $\mathcal{K}(r_0) = 0$, hence $r_0 - \mathcal{B}(r_0) = -\infty$. From the intermediate value theorem, there exists $r^* \in (r_0, \bar{r})$ such that $r^* - \mathcal{B}(r^*) = 0$. ☐

The analysis of the Huggett model is very similar and we only state the result.

Theorem 5.6. *Assume* $\underline{x} < B$ *and* (5.7)*, then there exists a solution to the Huggett model* (1.7)-(1.8) *with the equilibrium interest rate* r^* *such that* $0 < r^* < \rho$.

6 Numerical examples

We report on some numerical tests with the Aiyagari model, fixing

$$\begin{aligned}&\text{Discount}: \rho = 0.05,\ \text{income}: y_1 = 0.1,\ y_2 = 0.5,\\ &\text{Transition rates}: \lambda_1 = 0.4,\ \lambda_2 = 0.4,\ \text{Debt limit}: \underline{x} = -0.15.\\ &\text{TFP}: A = 0.95,\ \alpha = 0.35,\ \text{Depreciation}: \delta = 0.1.\end{aligned} \tag{6.1}$$

We considered four different cases. Hereafter, we enumerate the different cases and the related interest rate r^* at equilibrium:

- *Test 1:* $\gamma = 2$, $\psi = 0.8$, $r^* = 0.034$.
- *Test 2:* $\gamma = 2$, $\psi = 0.4$, $r^* = 0.0246$.
- *Test 3:* $\gamma = 4$, $\psi = 0.4$, $r^* = 0.018$.
- *Test 4:* $\gamma = 1.2$, $\psi = 0.4$, $r^* = 0.02737$.

It is important to notice that only Test 2 and Test 4 satisfy the assumption $\gamma\psi < 1$. Test 1 and Test 3 do not actually fall into the theoretical framework of the present paper. Yet, we observe that our numerical algorithms continue to perform well in these settings.

For comparison, in the CRRA case with $\gamma = 1/\psi = 2$ we obtain $r^* = 0.027942$.

Interpretations:

- Fixing γ, r^* increases as ψ increases. As agents are more willing to consume, the aggregate capital decreases.
- Fixing ψ, r^* decreases as γ increases. As agents are more risk averse, they favor more precautionary savings near $\underline{x}$.

We now plot the consumption and saving policies. For plotting the asset distributions, we truncated the upper range using a percentile-based threshold to avoid that the high values related to the Dirac mass hide the other ones. In the plots, we use solid lines for results from Test 1 and Test 3, dotted lines for Test 2 and Test 4.

We observe in Fig. 4 that, given the same ψ, agents in the model with higher risk aversion γ (Test 3) exhibit higher savings when their asset level x is close to the borrowing limit $\underline{x}$. This occurs despite the fact that the equilibrium r^* is substantially lower in Test 3 than in Test 4. Savings in Test 4 eventually exceed those in Test 3 as x moves away from $\underline{x}$. This pattern suggests that, near the borrowing constraint, the precautionary savings motive dominates the effect of interest rate differences, whereas the latter becomes more influential at higher asset levels.

7 Discussion and future works

In the near future, we plan to address the other situation, namely $\gamma\psi > 1$, when the agents favor early resolution of uncertainty. Many of the arguments contained in the present paper will have to be modified, and we plan to build on the existing literature on recursive utilities in discrete time.

Another direction of our future research concerns the analysis proposed in Section 5 and Appendix B in the limit situation when $r = \rho$, that is based on a second order expansion of the saving

Figure 1: Consumption with $\gamma = 2$: *Test 1* and *Test 2*

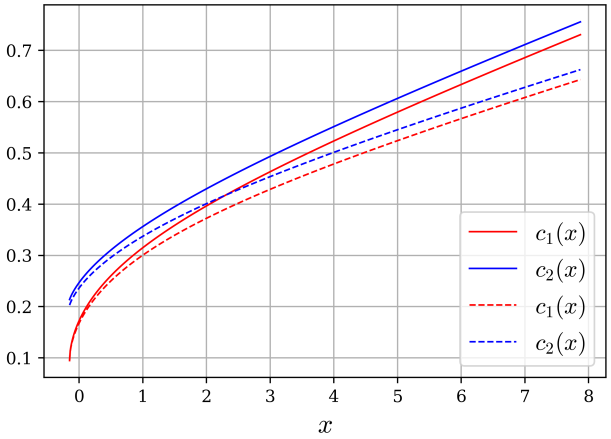


Figure 2: Consumption with $\psi = 0.4$: *Test 3* and *Test 4*

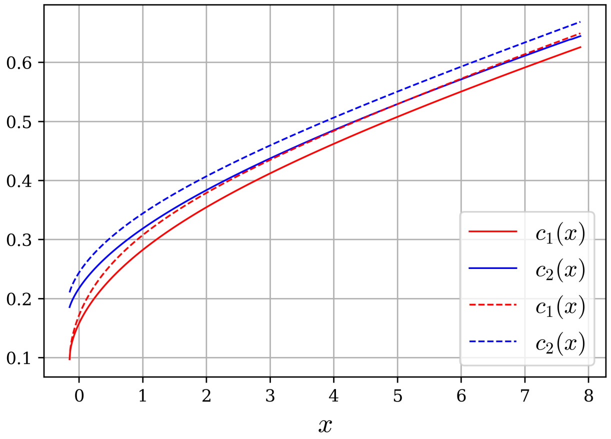


Figure 3: Saving with $\gamma = 2$: *Test 1* and *Test 2*

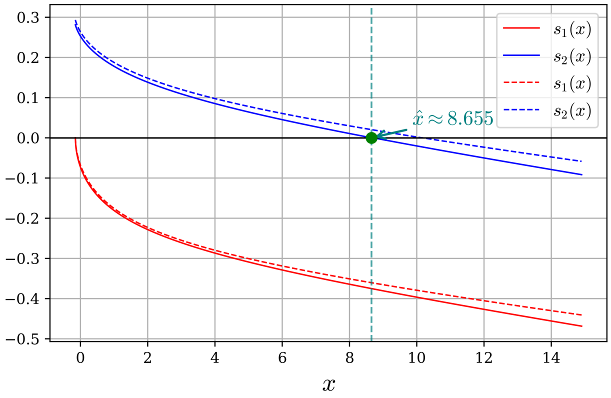


Figure 4: Saving with $\psi = 0.4$: *Test 3* and *Test 4*

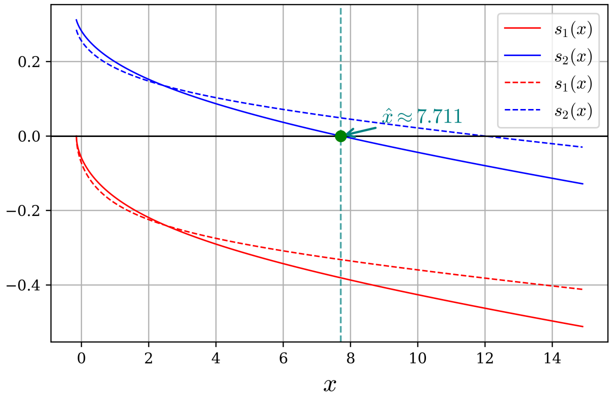


Figure 5: Asset distribution with $\gamma = 2$: *Test 1* and *Test 2*

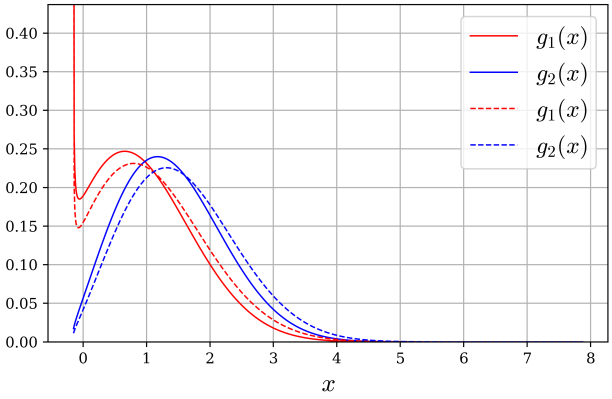
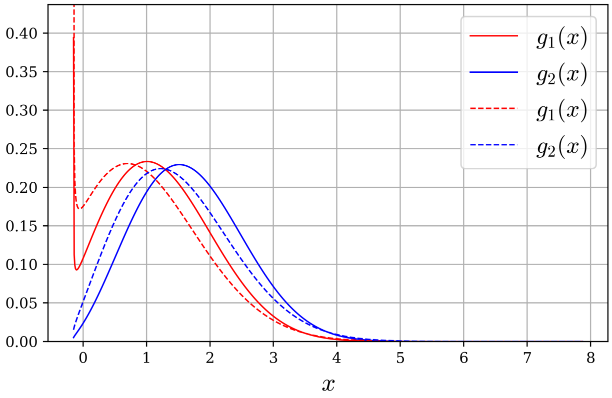


Figure 6: Asset distribution $\psi = 0.4$: *Test 3* and *Test 4*



policies as $x \to +\infty$. This method is specific to the two states income process in the present paper. It would be desirable to find a robust method that would work for general income processes. This may require using alternative arguments from the literature on the stability of Markov chains, see e.g. [31].

The present paper contains numerical results but, for brevity, we have chosen not to describe nor analyze the numerical methods that were used. We plan to address the numerical analysis of two different methods (finite differences and a semi-Lagrangian method). More precisely, we plan to investigate the convergence of the scheme (with the Barles-Souganidis theory, cf. [6]) and to study convergence rates as in [14].

Finally, the present paper considered models where there is no aggregate uncertainty impacting the economy globally. To consider aggregate uncertainty as in the Krusell-Smith model [28], one needs to study a master equation on the space of probability measures, cf. [13]. The numerical simulations of the master equation in continuous-time heterogeneous agent models have been addressed in the recent literature [7, 21]. It would be interesting to investigate the master equation arising in models with recursive utility.

Appendices

A Proof of the strong comparison principle

Proof of Theorem 3.3. We assume by contradiction that

$$\max_j \sup_x (\mathsf{u}_j(x) - \mathsf{v}_j(x)) = \delta > 0. \tag{A.1}$$

Step 1. First we consider the case when the sup in (A.1) is achieved at $\underline{x}$, i.e.

$$\max_j \sup_x (\mathsf{u}_j(x) - \mathsf{v}_j(x)) = \max_j (\mathsf{u}_j(\underline{x}) - \mathsf{v}_j(\underline{x})) = \delta. \tag{A.2}$$

The supersolution is defined only in $(\underline{x}, +\infty)$, but from its lower semicontinuity we can extend it to $\underline{x}$ with $\mathsf{v}_j(\underline{x}) = \liminf_{z \to \underline{x},\, z > \underline{x}} \mathsf{v}_j(z)$. There exists a sequence ζ_k such that

$$\mathsf{v}_j(\zeta_k) \to \mathsf{v}_j(\underline{x}) \quad \text{as} \quad \zeta_k \to \underline{x}, \quad j \in \{1, 2\}. \tag{A.3}$$

We denote $\epsilon_k = |\zeta_k - \underline{x}|$. Consider the function

$$\psi_k(j, x, z) = \mathsf{u}_j(x) - \mathsf{v}_j(z) - \frac{|x - z|^2}{\epsilon_k} - \left[\left(\frac{z - x}{\epsilon_k} - 1 \right)_- \right]^2 - |z - \underline{x}|^2. \tag{A.4}$$

Let ψ_k attain its maximum at (j_k, x_k, z_k). Since $\psi_k(j_k, x_k, z_k) \geq \psi_k(j_{\bar{j}_k}, x_k, z_k)$, we obtain

$$\mathsf{u}_{\bar{j}_k}(x_k) - \mathsf{u}_{j_k}(x_k) \leq \mathsf{v}_{\bar{j}_k}(z_k) - \mathsf{v}_{j_k}(z_k). \tag{A.5}$$

We now show

$$\psi_k(j_k, x_k, z_k) \geq \delta - o(1) > 0. \tag{A.6}$$

From $\left[\left(\frac{\zeta_k - \underline{x}}{\epsilon_k} - 1 \right)_- \right]^2 = 0$, we have

$$\psi_k(j_k, x_k, z_k) \geq \max_j \psi_k(j, \underline{x}, \zeta_k) = \max_j (\mathsf{u}_j(\underline{x}) - \mathsf{v}_j(\zeta_k)) - |\underline{x} - \zeta_k| - |\zeta_k - \underline{x}|^2.$$

From (A.3) we obtain $\max_j \psi_k(j, \underline{x}, \zeta_k) = \delta - o(1)$ and therefore (A.6). From $\psi_k(j_k, x_k, z_k) > 0$ and the boundedness of $\mathsf{u}_{j_k}(x_k)$ and $\mathsf{v}_{j_k}(z_k)$ there exists a constant $C > 0$ such that $\frac{|x_k - z_k|^2}{\epsilon_k} < C$, hence $x_k - z_k \to 0$ as $\epsilon_k \to 0$. From (A.6) we deduce

$$\liminf_{k\to+\infty}(\mathsf{u}_{j_k}(x_k) - \mathsf{v}_{j_k}(z_k)) \geq \liminf_{k\to+\infty} \psi_k(j_k, x_k, z_k) \geq \delta.$$

On the other hand, since $\mathsf{u}_{j_k}(x_k) - \mathsf{v}_{j_k}(z_k)$ is u.s.c.,

$$\limsup_{k\to+\infty}(\mathsf{u}_{j_k}(x_k) - \mathsf{v}_{j_k}(z_k)) \leq \max_j(\mathsf{u}_j(\underline{x}) - \mathsf{v}_j(\underline{x})) = \delta,$$

hence $\lim_{k\to+\infty}(\mathsf{u}_{j_k}(x_k) - \mathsf{v}_{j_k}(z_k)) = \delta$. We then obtain

$$\frac{|x_k - z_k|^2}{\epsilon_k} + \left[\left(\frac{z_k - x_k}{\epsilon_k} - 1\right)_-\right]^2 + |z_k - \underline{x}|^2 \to 0, \qquad \text{as} \quad \epsilon_k \to 0.$$

This gives $z_k - x_k \geq \epsilon_k - \epsilon_k o(1)$, which implies $z_k > \underline{x}$, hence we can use the supersolution property at z_k. We set

$$\Lambda_k = \frac{2}{\epsilon_k}\left(\frac{z_k - x_k}{\epsilon_k} - 1\right)_- .$$

Since $\psi_k(j_k, x, z)$ achieves its maximum at (x_k, z_k), we have

$$\begin{cases} \dfrac{\rho}{\theta}\mathsf{u}_{j_k}(x_k) \leq H\left(x_k, y_{j_k}, \mathsf{u}_{j_k}(x_k), \dfrac{2(x_k - z_k)}{\epsilon_k} + \Lambda_k\right) + \lambda_{j_k}\left(\mathsf{u}_{\bar{j}_k}(x_k) - \mathsf{u}_{j_k}(x_k)\right), \\ \dfrac{\rho}{\theta}\mathsf{v}_{j_k}(z_k) \geq H\left(z_k, y_{j_k}, \mathsf{v}_{j_k}(z_k), \dfrac{2(x_k - z_k)}{\epsilon_k} + \Lambda_k - 2(z_k - \underline{x})\right) + \lambda_{j_k}\left(\mathsf{v}_{\bar{j}_k}(z_k) - \mathsf{v}_{j_k}(z_k)\right). \end{cases} \tag{A.7}$$

From the coercivity of H and boundedness of v_{j_k}, it follows that $\frac{2(x_k - z_k)}{\epsilon_k} + \Lambda_k - 2(z_k - \underline{x})$ is positive and bounded. Subtracting the two inequalities in (A.7) and using (A.5), for k sufficiently large, we have

$$\begin{aligned} &\frac{\rho}{\theta}(\mathsf{u}_{j_k}(x_k) - \mathsf{v}_{j_k}(z_k)) \\ &\leq \underbrace{H\left(x_k, y_{j_k}, \mathsf{u}_{j_k}(x_k), \frac{2(x_k - z_k)}{\epsilon_k} + \Lambda_k\right) - H\left(x_k, y_{j_k}, \mathsf{v}_{j_k}(z_k), \frac{2(x_k - z_k)}{\epsilon_k} + \Lambda_k\right)}_{(I)<0} \\ &\quad + H\left(x_k, y_{j_k}, \mathsf{v}_{j_k}(z_k), \frac{2(x_k - z_k)}{\epsilon_k} + \Lambda_k\right) - H\left(z_k, y_{j_k}, \mathsf{v}_{j_k}(z_k), \frac{2(x_k - z_k)}{\epsilon_k} + \Lambda_k - 2(z_k - \underline{x})\right) \\ &\leq \underbrace{r(x_k - z_k)\left(\frac{2(x_k - z_k)}{\epsilon_k} + \Lambda_k\right)}_{(II)<0} + 2(rz_k + y_{j_k})(z_k - \underline{x}) \\ &\quad + \left[\underbrace{\frac{\rho^\psi}{\psi - 1}\left(\frac{2(x_k - z_k)}{\epsilon_k} + \Lambda_k\right)^{1-\psi} - \frac{\rho^\psi}{\psi - 1}\left(\frac{2(x_k - z_k)}{\epsilon_k} + \Lambda_k - 2(z_k - \underline{x})\right)^{1-\psi}}_{(III)<0}\right]((1-\gamma)\mathsf{v}_{j_k}(z_k))^{\frac{1-\gamma\psi}{1-\gamma}} \\ &\leq 2(rz_k + y_{j_k})(z_k - \underline{x}). \end{aligned} \tag{A.8}$$

In (A.8), (I) follows from (2.10) and $\lim_{k\to+\infty}(\mathsf{u}_{j_k}(x_k) - \mathsf{v}_{j_k}(z_k)) = \delta$, (II) follows from $z_k > x_k$ and (III) follows from the monotonicity of $(p^{1-\psi})/(\psi-1)$. We then obtain $\mathsf{u}_{j_k}(x_k) - \mathsf{v}_{j_k}(z_k) \to 0$ as $\epsilon_k \to 0$, in contradiction with (A.6). We have shown that (A.2) cannot hold.

Step 2. Now we consider the case when the supremum of $\mathsf{u}_j(x) - \mathsf{v}_j(x)$ is not achieved at $\underline{x}$, i.e.

$$\max_j \sup_x (\mathsf{u}_j(x) - \mathsf{v}_j(x)) = \delta > \max_j (\mathsf{u}_j(\underline{x}) - \mathsf{v}_j(\underline{x})). \tag{A.9}$$

Let us define the function $\mathtt{h}(x) = \frac{1}{2}\ln(1 + |x - \underline{x}|^2)$. From (A.9), one can show that for a sufficiently small $\beta > 0$, we can find $0 < \delta_2 \le \delta$ such that

$$\max_j \sup_x (\mathsf{u}_j(x) - \mathsf{v}_j(x) - \beta\mathtt{h}(x)) > \delta_2 > \max_j (\mathsf{u}_j(\underline{x}) - \mathsf{v}_j(\underline{x})). \tag{A.10}$$

We now consider the function

$$\Phi(j, x, z) = \mathsf{u}_j(x) - \mathsf{v}_j(z) - \frac{|x-z|^2}{\epsilon} - \beta\mathtt{h}(x). \tag{A.11}$$

Since $\Phi(j, x, z)$ is u.s.c., and $\beta\mathtt{h}(x) \to +\infty$ as $x \to +\infty$, we can assume that $\Phi(j, x, z)$ achieves its maximum at some $(j_\epsilon, x_\epsilon, z_\epsilon)$ such that $x_\epsilon, z_\epsilon < +\infty$ if $\beta > 0$. Now we fix β such that (A.10) holds and

$$\beta(\rho + y_2) < \frac{\delta_2\rho}{4\theta}, \tag{A.12}$$

where θ is defined in Table 1. It is obvious that $\Phi(j_\epsilon, x_\epsilon, z_\epsilon) \ge \max_j \sup_x (\mathsf{u}_j(x) - \mathsf{v}_j(x) - \beta\mathtt{h}(x))$, hence (A.10) yields

$$\Phi(j_\epsilon, x_\epsilon, z_\epsilon) > \delta_2 \text{ for all } \varepsilon.$$

The boundedness of u_j and v_j yields

$$|x_\epsilon - z_\epsilon| \to 0 \quad \text{as} \quad \epsilon \to 0.$$

Classically, we then infer that

$$\frac{|x_\epsilon - z_\epsilon|^2}{\epsilon} \to 0 \quad \text{as} \quad \epsilon \to 0.$$

We then extract subsequences x_{ϵ_k} and z_{ϵ_k} that both converge to some x^* and such that j_{ϵ_k} converges to j^*. From the semicontinuity of u_j and v_j, with (A.12), we get

$$\mathsf{u}_{j^*}(x^*) - \mathsf{v}_{j^*}(x^*) \ge \limsup_k (\mathsf{u}_{j_{\epsilon_k}}(x_{\epsilon_k}) - \mathsf{v}_{j_{\epsilon_k}}(z_{\epsilon_k})) \ge \Phi(j_{\epsilon_k}, x_{\epsilon_k}, z_{\epsilon_k}) > \delta_2. \tag{A.13}$$

It follows from (A.10) and (A.13) that $x^* > \underline{x}$. For brevity we write the sequence (j_k, x_k, y_k) instead of $(j_{\epsilon_k}, x_{\epsilon_k}, y_{\epsilon_k})$. From Theorem 3.1 we obtain

$$
\begin{aligned}
&\frac{\rho}{\theta}(\mathsf{u}_{j_k}(x_k) - \mathsf{v}_{j_k}(z_k)) \\
&\le \underbrace{H\left(x_k, y_{j_k}, \mathsf{u}_{j_k}(x_k), \frac{2(x_k - z_k)}{\epsilon_k} + \beta D\mathtt{h}(x_k)\right) - H\left(x_k, y_{j_k}, \mathsf{v}_{j_k}(z_k), \frac{2(x_k - z_k)}{\epsilon_k} + \beta D\mathtt{h}(x_k)\right)}_{(I)<0} \\
&\quad + H\left(x_k, y_{j_k}, \mathsf{v}_{j_k}(z_k), \frac{2(x_k - z_k)}{\epsilon_k} + \beta D\mathtt{h}(x_k)\right) - H\left(z_k, y_{j_k}, \mathsf{v}_{j_k}(z_k), \frac{2(x_k - z_k)}{\epsilon_k}\right) \\
&\le r(x_k - z_k)\left(\frac{2(x_k - z_k)}{\epsilon_k}\right) + \underbrace{\beta(r x_k + y_{j_k}) D\mathtt{h}(x_k)}_{(II)} \\
&\quad + \left[\underbrace{\frac{\rho^\psi}{\psi - 1}\left(\frac{2(x_\epsilon - z_\epsilon)}{\epsilon} + \beta D\mathtt{h}(x_k)\right)^{1-\psi} - \frac{\rho^\psi}{\psi - 1}\left(\frac{2(x_k - z_k)}{\epsilon_k}\right)^{1-\psi}}_{(III)<0}\right]((1-\gamma)\mathsf{v}_{j_k}(z_k))^{\frac{1-\gamma\psi}{1-\gamma}} \\
&\le \frac{2\rho(x_k - z_k)^2}{\epsilon_k} + \frac{\delta_2 \rho}{4\theta}.
\end{aligned}
\tag{A.14}
$$

In (A.14), (I) and (III) are dealt with in the same way as in (A.8), cf. *Step 1*. It follows from (A.12), with $D\mathtt{h}(x_k) < 1$ and $x_k D\mathtt{h}(x_k) \le 1$, that $(II) < \frac{\delta_2 \rho}{4\theta}$. Passing to the limit yields $\mathsf{u}_{j^*}(x^*) - \mathsf{v}_{j^*}(z^*) < \delta_2$, in contradiction with (A.13).

Combining *Step 1* and *Step 2*, we have shown that (A.1) cannot hold, leading to the desired result. $\square$

B Asymptotic analysis as $x \to +\infty$ in the case $r = \rho$

Proof of Theorem 5.2. From Theorem 3.16 and (5.1), we know that if $r = \rho$ then

$$\frac{(\rho x + y_1)^{1-\gamma}}{(1-\gamma)} \le v_1(x) < v_2(x) \le \frac{(\rho x + y_2)^{1-\gamma}}{(1-\gamma)} \quad \forall x > \underline{x}. \tag{B.1}$$

From $s_1(x) < 0$ when $x > \underline{x}$, we deduce

$$\rho x + y_1 < \rho^\psi (Dv_1(x))^{-\psi}((1-\gamma)v_1(x))^{\frac{1-\gamma\psi}{1-\gamma}} \le \rho^\psi (Dv_1(x))^{-\psi}(\rho x + y_2)^{1-\gamma\psi},$$

$$0 < Dv_1(x) < \rho(\rho x + y_1)^{-\gamma}\left(\frac{\rho x + y_2}{\rho x + y_1}\right)^{1-\gamma\psi}. \tag{B.2}$$

Similarly,

$$Dv_2(x) > \rho(\rho x + y_2)^{-\gamma}\left(\frac{\rho x + y_1}{\rho x + y_2}\right)^{1-\gamma\psi}. \tag{B.3}$$

We then proceed in 2 steps.
Step 1: From (B.1), (B.2) and (B.3), we may look for a first order expansion of v_j in the form

$$v_j(x) = \frac{(\rho x + y_j)^{1-\gamma}}{(1-\gamma)} + \boldsymbol{z}_j(x).$$

with

$$\boldsymbol{z}_j(x) = O(x^{-\gamma}) \quad \text{and} \quad D\boldsymbol{z}_j(x) = O(x^{-1-\gamma}).$$

From

$$\frac{(\rho x + y_2)^{1-\gamma}}{(1-\gamma)} - \frac{(\rho x + y_1)^{1-\gamma}}{(1-\gamma)} - (\rho x + y_1)^{-\gamma}(y_2 - y_1) = O(x^{-1-\gamma}),$$

we infer

$$\begin{aligned}
&\left(\frac{\rho}{\theta} + \lambda_j\right) v_j(x) - \lambda_j v_{\bar{j}}(x) \\
&= \frac{\rho\,(\rho x + y_j)^{1-\gamma}}{1-\psi^{-1}} + \lambda_j\,(\rho x + y_j)^{-\gamma}(y_j - y_{\bar{j}}) + \left(\frac{\rho}{\theta} + \lambda_1\right) \boldsymbol{z}_j(x) - \lambda_j \boldsymbol{z}_{\bar{j}}(x) + O(x^{-1-\gamma}).
\end{aligned}$$

On the other hand, from (2.4) we deduce

$$\begin{aligned}
&H(x, y_j, v_j(x), Dv_j(x)) \\
&= (\rho x + y_j) Dv_j(x) + \frac{\rho^{\psi}}{\psi - 1}\,(Dv_j(x))^{1-\psi}\,((1-\gamma)v_j(x))^{\frac{1-\gamma\psi}{1-\gamma}} \\
&= \rho\,(\rho x + y_j)^{1-\gamma} + (\rho x + y_j) D\boldsymbol{z}_j(x) + \underbrace{\frac{\rho^{\psi}}{\psi - 1}\left(\rho\,(\rho x + y_j)^{-\gamma} + D\boldsymbol{z}_j(x)\right)^{1-\psi}((1-\gamma)v_j(x))^{\frac{1-\gamma\psi}{1-\gamma}}}_{(I)} \\
&= \frac{\rho\,(\rho x + y_j)^{1-\gamma}}{1-\psi^{-1}} + \left(\frac{1}{\theta} - 1\right)\rho \boldsymbol{z}_j(x) + O(x^{-1-\gamma}).
\end{aligned} \tag{B.4}$$

To obtain the expansion for (I), we observe

$$((1-\gamma)v_j(x))^{\frac{1-\gamma\psi}{1-\gamma}} = (\rho x + y_j)^{1-\gamma\psi}\left(1 + (1-\gamma)\frac{\boldsymbol{z}_j(x)}{(\rho x + y_j)^{1-\gamma}}\right)^{\frac{1-\gamma\psi}{1-\gamma}},$$

then, expanding to the order $O(x^{-1-\gamma})$, we obtain

$$\begin{aligned}
&\frac{\rho^{\psi}}{\psi - 1}\left(\rho\,(\rho x + y_j)^{-\gamma} + D\boldsymbol{z}_j(x)\right)^{1-\psi}(\rho x + y_j)^{1-\gamma\psi}\left(1 + (1-\gamma)\frac{\boldsymbol{z}_j(x)}{(\rho x + y_j)^{1-\gamma}}\right)^{\frac{1-\gamma\psi}{1-\gamma}} \\
&\sim \frac{\rho}{\psi - 1}\,(\rho x + y_j)^{1-\gamma}\left(1 + \rho^{-1}(\rho x + y_j)^{\gamma} D\boldsymbol{z}_j(x)\right)^{1-\psi}\left(1 + (1-\gamma\psi)\frac{\boldsymbol{z}_j(x)}{(\rho x + y_j)^{1-\gamma}}\right) \\
&\sim \frac{\rho\,(\rho x + y_j)^{1-\gamma}}{\psi - 1} - (\rho x + y_j)\,D\boldsymbol{z}_j(x) + \underbrace{\frac{1-\gamma\psi}{\psi - 1}}_{=1/\theta - 1}\rho \boldsymbol{z}_j(x).
\end{aligned}$$

From the HJB equation for v_j we obtain

$$(\rho + \lambda_j)\,\boldsymbol{z}_j(x) - \lambda_j \boldsymbol{z}_{\bar{j}}(x) = \lambda_j\,(\rho x + y_j)^{-\gamma}(y_{\bar{j}} - y_j) + O(x^{-1-\gamma}). \tag{B.5}$$

Solving (B.5) up to terms of order $O(x^{-\gamma})$, we obtain that $\boldsymbol{z}_j(x) \sim \widehat{\boldsymbol{z}}_j(x)$ as $x \to +\infty$, where

$$\widehat{\boldsymbol{z}}_j(x) = \frac{\lambda_j\,(\rho x + y_j)^{-\gamma}(y_{\bar{j}} - y_j)}{\rho + \lambda_j + \lambda_{\bar{j}}}. \tag{B.6}$$

Note that

$$D\widehat{\boldsymbol{z}}_j(x) = -\frac{\rho\gamma\lambda_j\left(\rho x + y_j\right)^{-1-\gamma}\left(y_{\bar{j}} - y_j\right)}{\rho + \lambda_j + \lambda_{\bar{j}}}. \tag{B.7}$$

Step 2: Second order expansion. Let us set

$$v_j(x) = \frac{(\rho x + y_j)^{1-\gamma}}{(1-\gamma)} + \widehat{\boldsymbol{z}}_j(x) + \boldsymbol{q}_j(x),$$

where, from the previous step, $\boldsymbol{q}_j(x) = O(x^{-1-\gamma})$ as $x \to +\infty$. Unfortunately, we do not have yet any better information on $D\boldsymbol{q}_j(x)$ than the estimate $D\boldsymbol{q}_j(x) = O(x^{-1-\gamma})$.

The HJB equation for $v_j(x)$ becomes

$$\begin{aligned}
&\frac{\rho}{\theta}\left(\frac{(\rho x + y_j)^{1-\gamma}}{(1-\gamma)} + \widehat{\boldsymbol{z}}_j(x) + \boldsymbol{q}_j(x)\right)\\
=\ &(\rho x + y_j)\left(\rho\left(\rho x + y_j\right)^{-\gamma} + D\widehat{\boldsymbol{z}}_j(x) + D\boldsymbol{q}_j(x)\right)\\
&+\underbrace{\frac{\rho}{\psi-1}\left(\rho x + y_j\right)^{1-\gamma}\left(1 + \rho^{-1}\left(\rho x + y_j\right)^{\gamma}\left(D\widehat{\boldsymbol{z}}_j(x) + D\boldsymbol{q}_j(x)\right)\right)^{1-\psi}\left(1 + (1-\gamma)\frac{\widehat{\boldsymbol{z}}_j(x) + \boldsymbol{q}_j(x)}{(\rho x + y_j)^{1-\gamma}}\right)^{\frac{1-\gamma\psi}{1-\gamma}}}_{(I)}\\
&+\lambda_j\left(\left(\frac{(\rho x + y_{\bar{j}})^{1-\gamma}}{(1-\gamma)} + \widehat{\boldsymbol{z}}_{\bar{j}}(x) + \boldsymbol{q}_{\bar{j}}(x)\right) - \left(\frac{(\rho x + y_j)^{1-\gamma}}{(1-\gamma)} + \widehat{\boldsymbol{z}}_j(x) + \boldsymbol{q}_j(x)\right)\right).
\end{aligned} \tag{B.8}$$

We aim at simplifying (B.8) by using the a priori information on the behavior of $\boldsymbol{q}_j$ at infinity. We start with the term (I): we observe that

$$\begin{aligned}
&\left(1 + \frac{(\rho x + y_j)^{\gamma}}{\rho}(D\widehat{\boldsymbol{z}}_j(x) + D\boldsymbol{q}_j(x))\right)^{1-\psi}\\
=&1 + \frac{1-\psi}{\rho}(\rho x + y_j)^{\gamma}(D\widehat{\boldsymbol{z}}_j(x) + D\boldsymbol{q}_j(x)) - \frac{(1-\psi)\psi}{2\rho^2}(\rho x + y_j)^{2\gamma}(D\widehat{\boldsymbol{z}}_j(x) + D\boldsymbol{q}_j(x))^2 + O(x^{-3}),
\end{aligned}$$

and that

$$\begin{aligned}
&\left(1 + (1-\gamma)(\rho x + y_j)^{\gamma-1}(\widehat{\boldsymbol{z}}_j(x) + \boldsymbol{q}_j(x))\right)^{\frac{1-\gamma\psi}{1-\gamma}}\\
=&1 + (1-\gamma\psi)(\rho x + y_j)^{\gamma-1}(\widehat{\boldsymbol{z}}_j(x) + \boldsymbol{q}_j(x)) + \frac{\gamma}{2}(1-\gamma\psi)(1-\psi)(\rho x + y_j)^{2\gamma-2}\widehat{\boldsymbol{z}}_j^2(x) + O(x^{-3}).
\end{aligned}$$

Hence the term (I) in (B.8) satisfies

$$\begin{aligned}
(I) = &\frac{\rho\left(\rho x + y_j\right)^{1-\gamma}}{\psi-1}\cdot\\
&\left(\begin{array}{l}
1 + \dfrac{1-\psi}{\rho}(\rho x + y_j)^{\gamma}(D\widehat{\boldsymbol{z}}_j(x) + D\boldsymbol{q}_j(x)) + (1-\gamma\psi)(\rho x + y_j)^{\gamma-1}(\widehat{\boldsymbol{z}}_j(x) + \boldsymbol{q}_j(x))\\
-\dfrac{(1-\psi)\psi}{2\rho^2}(\rho x + y_j)^{2\gamma}(D\widehat{\boldsymbol{z}}_j(x) + D\boldsymbol{q}_j(x))^2 + \dfrac{\gamma}{2}(1-\gamma\psi)(1-\psi)(\rho x + y_j)^{2\gamma-2}\widehat{\boldsymbol{z}}_j^2(x)\\
+\dfrac{1-\psi}{\rho}(1-\gamma\psi)(\rho x + y_j)^{2\gamma-1}(\widehat{\boldsymbol{z}}_j(x) + \boldsymbol{q}_j(x))(D\widehat{\boldsymbol{z}}_j(x) + D\boldsymbol{q}_j(x)) + O(x^{-3})
\end{array}\right)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\rho}{\psi-1}\left(\rho x+y_j\right)^{1-\gamma} + \left(\frac{\rho}{\theta}-\rho\right)\left(\widehat{\boldsymbol{z}}_j(x)+\boldsymbol{q}_j(x)\right) - \left(\rho x+y_j\right)\left(D\widehat{\boldsymbol{z}}_j(x)+D\boldsymbol{q}_j(x)\right) \\
&\quad + \underbrace{\frac{\psi}{2\rho}\left(\rho x+y_j\right)^{\gamma+1}|D\widehat{\boldsymbol{z}}_j(x)|^2}_{(I1)} - \underbrace{\left(1-\gamma\psi\right)\left(\rho x+y_j\right)^{\gamma}\widehat{\boldsymbol{z}}_j(x)D\widehat{\boldsymbol{z}}_j(x)}_{(I2)} \\
&\quad - \underbrace{\frac{\rho(1-\gamma\psi)\gamma}{2}\left(\rho x+y_j\right)^{\gamma-1}|\widehat{\boldsymbol{z}}_j(x)|^2}_{(I3)} \\
&\quad + \frac{\psi\left(\rho x+y_j\right)^{\gamma+1}}{2\rho}\big(|D\boldsymbol{q}_j(x)|^2+2D\widehat{\boldsymbol{z}}_j(x)D\boldsymbol{q}_j(x)\big) - \left(1-\gamma\psi\right)\left(\rho x+y_j\right)^{\gamma}\widehat{\boldsymbol{z}}_j(x)D\boldsymbol{q}_j(x) + O(x^{-\gamma-2}).
\end{aligned}
$$

From (B.6) and (B.7), we obtain

$$
(I1)-(I2)-(I3) = \frac{\rho\gamma}{2}\left(\frac{\lambda_1}{\rho+\lambda_1+\lambda_2}\right)^2\left(\rho x+y_1\right)^{-1-\gamma}\left(y_2-y_1\right)^2.
$$

We also notice that $\widehat{\boldsymbol{z}}_1$ satisfies the equation

$$
\left(\rho+\lambda_1\right)\widehat{\boldsymbol{z}}_1(x)-\lambda_1\widehat{\boldsymbol{z}}_2(x) = \lambda_1\left(\rho x+y_1\right)^{-\gamma}\left(y_2-y_1\right) + \frac{\lambda_1\lambda_2\gamma(\rho x+y_1)^{-1-\gamma}(y_2-y_1)^2}{\rho+\lambda_1+\lambda_2} + O(x^{-2-\gamma}),
$$

and we observe that

$$
\frac{\left(\rho x+y_2\right)^{1-\gamma}}{(1-\gamma)} - \frac{\left(\rho x+y_1\right)^{1-\gamma}}{(1-\gamma)} = \left(\rho x+y_1\right)^{-\gamma}\left(y_2-y_1\right) - \frac{\gamma}{2}\left(\rho x+y_1\right)^{-1-\gamma}\left(y_2-y_1\right)^2 + O(x^{-2-\gamma}).
$$

By using the equalities above, (B.8) becomes:

$$
\begin{aligned}
&\left(\rho+\lambda_1\right)\boldsymbol{q}_1-\lambda_1\boldsymbol{q}_2-\frac{\psi}{2\rho}\left(\rho x+y_1\right)^{\gamma+1}\big(|D\boldsymbol{q}_1(x)|^2+2D\widehat{\boldsymbol{z}}_1(x)D\boldsymbol{q}_1(x)\big) \\
&\quad + \left(1-\gamma\psi\right)\left(\rho x+y_1\right)^{\gamma}\widehat{\boldsymbol{z}}_1(x)D\boldsymbol{q}_1(x) \\
&= -\frac{\lambda_1\gamma}{2}\left(\rho x+y_1\right)^{-1-\gamma}\left(y_2-y_1\right)^2\left(1-\frac{\rho\lambda_1}{(\rho+\lambda_1+\lambda_2)^2}-\frac{2\lambda_2}{\rho+\lambda_1+\lambda_2}\right)+O(x^{-2-\gamma}),
\end{aligned}
\tag{B.9}
$$

and

$$
\begin{aligned}
&\left(\rho+\lambda_2\right)\boldsymbol{q}_2-\lambda_2\boldsymbol{q}_1-\frac{\psi}{2\rho}\left(\rho x+y_2\right)^{\gamma+1}\big(|D\boldsymbol{q}_2(x)|^2+2D\widehat{\boldsymbol{z}}_2(x)D\boldsymbol{q}_2(x)\big) \\
&\quad + \left(1-\gamma\psi\right)\left(\rho x+y_2\right)^{\gamma}\widehat{\boldsymbol{z}}_2(x)D\boldsymbol{q}_2(x) \\
&= -\frac{\lambda_2\gamma}{2}\left(\rho x+y_2\right)^{-1-\gamma}\left(y_2-y_1\right)^2\left(1-\frac{\rho\lambda_2}{(\rho+\lambda_1+\lambda_2)^2}-\frac{2\lambda_1}{\rho+\lambda_1+\lambda_2}\right)+O(x^{-2-\gamma}).
\end{aligned}
\tag{B.10}
$$

We then rescale $\boldsymbol{q}_j$ as follows: $\boldsymbol{q}_j(x) = -(\rho x+y_j)^{-\gamma-1}Q_j(x)$. The a priori information on $\boldsymbol{q}_j$ yields that Q_j are smooth functions and that $Q_j(x)=O(1)$ and $DQ_j(x)=O(1)$ as $x\to+\infty$. We then obtain from (B.9) and (B.10) that

$$
\begin{aligned}
&\left(\rho+\lambda_1\right)Q_1-\lambda_1Q_2+\frac{\psi}{2\rho}|DQ_1(x)|^2+\frac{\lambda_1}{\rho+\lambda_1+\lambda_2}(y_2-y_1)DQ_1 \\
&= \frac{\lambda_1\gamma}{2}(y_2-y_1)^2\left(1-\frac{\rho\lambda_1}{(\rho+\lambda_1+\lambda_2)^2}-\frac{2\lambda_2}{\rho+\lambda_1+\lambda_2}\right)+O(x^{-1}),
\end{aligned}
\tag{B.11}
$$

and

$$\begin{aligned}&(\rho+\lambda_2)\,Q_2-\lambda_2 Q_1+\frac{\psi}{2\rho}|DQ_2(x)|^2+\frac{\lambda_2}{\rho+\lambda_1+\lambda_2}(y_1-y_2)DQ_2\\&=\frac{\lambda_2\gamma}{2}(y_1-y_2)^2\left(1-\frac{\rho\lambda_2}{(\rho+\lambda_1+\lambda_2)^2}-\frac{2\lambda_1}{\rho+\lambda_1+\lambda_2}\right)+O(x^{-1}),\end{aligned}\tag{B.12}$$

We notice that the unique smooth and bounded solution of the following system of Hamilton-Jacobi equations posed in the full lines $\mathbb{R}\times\mathbb{R}$:

$$\begin{aligned}&(\rho+\lambda_1)\,\widehat{Q}_1-\lambda_1\widehat{Q}_2+\frac{\psi}{2\rho}|D\widehat{Q}_1(x)|^2+\frac{\lambda_1(y_2-y_1)}{\rho+\lambda_1+\lambda_2}D\widehat{Q}_1(x)\\&=\frac{\lambda_1\gamma}{2}(y_2-y_1)^2\left(1-\frac{\rho\lambda_1}{(\rho+\lambda_1+\lambda_2)^2}-\frac{2\lambda_2}{\rho+\lambda_1+\lambda_2}\right),\end{aligned}$$

$$\begin{aligned}&(\rho+\lambda_2)\,\widehat{Q}_2-\lambda_2\widehat{Q}_1+\frac{\psi}{2\rho}|D\widehat{Q}_2(x)|^2+\frac{\lambda_2(y_1-y_2)}{\rho+\lambda_1+\lambda_2}D\widehat{Q}_2(x)\\&=\frac{\lambda_2\gamma}{2}(y_1-y_2)^2\left(1-\frac{\rho\lambda_2}{(\rho+\lambda_1+\lambda_2)^2}-\frac{2\lambda_1}{\rho+\lambda_1+\lambda_2}\right).\end{aligned}$$

is given by the constants

$$\begin{aligned}\widehat{Q}_1&=\frac{\gamma\lambda_1(y_2-y_1)^2}{2(\rho+\lambda_1+\lambda_2)}\left(1-\frac{\lambda_1\lambda_2+\lambda_2^2+\rho\lambda_1}{(\rho+\lambda_1+\lambda_2)^2}\right),\\\widehat{Q}_2&=\frac{\gamma\lambda_2(y_2-y_1)^2}{2(\rho+\lambda_1+\lambda_2)}\left(1-\frac{\lambda_1\lambda_2+\lambda_1^2+\rho\lambda_2}{(\rho+\lambda_1+\lambda_2)^2}\right).\end{aligned}$$

This indicates that when $x\to+\infty$, $\boldsymbol{q}_j(x)\sim\widehat{\boldsymbol{q}}_j(x)$ (for brevity, we omit the proof of this point), where

$$\widehat{\boldsymbol{q}}_1(x)=-\frac{\gamma\lambda_1(y_2-y_1)^2}{2(\rho+\lambda_1+\lambda_2)}\left(1-\frac{\lambda_1\lambda_2+\lambda_2^2+\rho\lambda_1}{(\rho+\lambda_1+\lambda_2)^2}\right)(\rho x+y_1)^{-1-\gamma},\tag{B.13}$$

$$\widehat{\boldsymbol{q}}_2(x)=-\frac{\gamma\lambda_2(y_2-y_1)^2}{2(\rho+\lambda_1+\lambda_2)}\left(1-\frac{\lambda_1\lambda_2+\lambda_1^2+\rho\lambda_2}{(\rho+\lambda_1+\lambda_2)^2}\right)(\rho x+y_2)^{-1-\gamma}.\tag{B.14}$$

Moreover, injecting the information that $Q_j(x)\sim\widehat{Q}_j(x)$ as $x\to+\infty$ into (B.11)-(B.12), we see that as $x\to\infty$,

$$\begin{aligned}\frac{\psi}{2\rho}|DQ_1(x)|^2+\frac{\lambda_1}{\rho+\lambda_1+\lambda_2}(y_2-y_1)DQ_1(x)&=o(1),\\\frac{\psi}{2\rho}|DQ_2(x)|^2+\frac{\lambda_2}{\rho+\lambda_1+\lambda_2}(y_1-y_2)DQ_2(x)&=o(1).\end{aligned}$$

Since Q_1 and Q_2 are smooth and tend to constants at $+\infty$, the latter system of equations implies that $DQ_j(x)=o(1)$ as $x\to+\infty$, $j=1,2$. This implies that $D\boldsymbol{q}_j(x)\sim D\widehat{\boldsymbol{q}}_j(x)$ as $x\to\infty$, where

$$\begin{aligned}D\widehat{\boldsymbol{q}}_1(x)&=\frac{\rho\gamma(1+\gamma)\lambda_1(y_2-y_1)^2}{2(\rho+\lambda_1+\lambda_2)}\left(1-\frac{\lambda_1\lambda_2+\lambda_2^2+\rho\lambda_1}{(\rho+\lambda_1+\lambda_2)^2}\right)(\rho x+y_1)^{-2-\gamma},\\D\widehat{\boldsymbol{q}}_2(x)&=\frac{\rho\gamma(1+\gamma)\lambda_2(y_2-y_1)^2}{2(\rho+\lambda_1+\lambda_2)}\left(1-\frac{\lambda_1\lambda_2+\lambda_1^2+\rho\lambda_2}{(\rho+\lambda_1+\lambda_2)^2}\right)(\rho x+y_2)^{-2-\gamma}.\end{aligned}$$

We now derive the asymptotic behavior of the optimal consumption as $x \to +\infty$:

$$\begin{aligned}
&c_1(x) \\
&= \rho^{\psi} \left(\rho\left(\rho x + y_1\right)^{-\gamma} + D\widehat{\boldsymbol{z}}_1(x) + D\boldsymbol{q}_1(x)\right)^{-\psi} \left((1-\gamma)v_1(x)\right)^{\frac{1-\gamma\psi}{1-\gamma}} \\
&= \left(\rho x + y_1\right)\left[1 - \frac{\psi\left(\rho x + y_1\right)^{\gamma}}{\rho}\left(D\widehat{\boldsymbol{z}}_1(x) + D\widehat{\boldsymbol{q}}_1(x)\right) + \frac{\left(\rho x + y_1\right)^{2\gamma}\psi(1+\psi)}{2\rho^2}|D\widehat{\boldsymbol{z}}_1(x)|^2\right] \\
&\quad \cdot \left[1 + (1-\gamma\psi)\frac{\widehat{\boldsymbol{z}}_1(x) + \widehat{\boldsymbol{q}}_1(x)}{\left(\rho x + y_1\right)^{1-\gamma}} + \frac{(1-\gamma\psi)\gamma(1-\psi)}{2\left(\rho x + y_1\right)^{2-2\gamma}}|\widehat{\boldsymbol{z}}_1(x)|^2\right] + o((\rho x + y_1)^{-1}) \\
&= \left(\rho x + y_1\right) + \frac{\lambda_1(y_2 - y_1)}{\rho + \lambda_1 + \lambda_2} + \frac{\gamma(1+\psi)\lambda_1^2(y_2 - y_1)^2}{2(\rho + \lambda_1 + \lambda_2)^2}\left(\rho x + y_1\right)^{-1} \\
&\quad - \frac{\gamma(1+\psi)\lambda_1(y_2 - y_1)^2}{2(\rho + \lambda_1 + \lambda_2)}\left(1 - \frac{\lambda_1\lambda_2 + \lambda_2^2 + \rho\lambda_1}{(\rho + \lambda_1 + \lambda_2)^2}\right)\left(\rho x + y_1\right)^{-1} + o((\rho x + y_1)^{-1}).
\end{aligned}$$

The calculation for c_2 is similar. We can then deduce the expansion of s_j in (5.5). ☐

Proof of Theorem 5.3. Suppose there exists an invariant measure $m = (m_1, m_2)$ when $r = \rho$, then it has densities g_j defined on $[\underline{x}, +\infty)$, $j \in \{2, 1\}$, given by (4.8) and (4.9). We write (5.5) as

$$\begin{aligned}
s_1(x) = \frac{\lambda_1(y_1 - y_2)}{\rho + \lambda_1 + \lambda_2} + \frac{\gamma(1+\psi)\lambda_1(y_2 - y_1)^2}{2(\rho + \lambda_1 + \lambda_2)}&\left[\underbrace{1 - \frac{\lambda_1\lambda_2 + \lambda_2^2 + \rho\lambda_1}{(\rho + \lambda_1 + \lambda_2)^2} - \frac{\lambda_1}{(\rho + \lambda_1 + \lambda_2)}}_{(I)}\right]\left(\rho x + y_1\right)^{-1} \\
&+ o((\rho x + y_1)^{-1}), \\
s_2(x) = \frac{\lambda_2(y_2 - y_1)}{\rho + \lambda_1 + \lambda_2} + \frac{\gamma(1+\psi)\lambda_2(y_2 - y_1)^2}{2(\rho + \lambda_1 + \lambda_2)}&\left[\underbrace{1 - \frac{\lambda_1\lambda_2 + \lambda_1^2 + \rho\lambda_2}{(\rho + \lambda_1 + \lambda_2)^2} - \frac{\lambda_2}{(\rho + \lambda_1 + \lambda_2)}}_{(II)}\right]\left(\rho x + y_2\right)^{-1} \\
&+ o((\rho x + y_2)^{-1}).
\end{aligned}$$

We notice that in the equations for $s_1(x)$ and $s_2(x)$,

$$(I) + (II) = 1 + \frac{\rho}{(\rho + \lambda_1 + \lambda_2)} - \frac{\lambda_1 + \lambda_2}{(\rho + \lambda_1 + \lambda_2)} = \frac{2\rho}{(\rho + \lambda_1 + \lambda_2)},$$

it follows that as $x \to +\infty$,

$$\frac{\lambda_2 s_1(x) + \lambda_1 s_2(x)}{-s_1(x)s_2(x)} = \frac{\rho\gamma(1+\psi)}{\rho x + y_1} + o((\rho x + y_1)^{-1}).$$

With the formulas (4.8) and (4.9), we obtain $g_2(x)$ and $g_1(x)$ tend to $+\infty$ as $x \to +\infty$, in contradiction with the fact that m_1 and m_2 have finite mass. ☐

C Asymptotic analysis near $x = \underline{x}$

We now consider the behavior of s_1 near $\underline{x}$. From (2.7) and $s_1(\underline{x}) = 0$, we have

$$Dv_1(\underline{x}) = \rho(r\underline{x} + y_1)^{-\psi^{-1}}((1-\gamma)v_1(\underline{x}))^{\frac{\psi^{-1}-\gamma}{1-\gamma}}. \tag{C.1}$$

From Theorem 3.17, we can differentiate the HJB equation for v_1 at $x > \underline{x}$ and pass to the limit $x \to \underline{x}$,

$$(\zeta - r)Dv_1(\underline{x}) + \lambda_1(Dv_1(\underline{x}) - Dv_2(\underline{x})) = \lim_{x\to\underline{x}} s_1(x)D^2v_1(x) + H_v(\underline{x}, y_1, v_1(\underline{x}), Dv_1(\underline{x}))Dv_1(\underline{x}).$$

Moreover, $\lim_{x\to\underline{x}} s_1(x)D^2v_1(x) > 0$. We denote the constant

$$\varkappa := (\zeta - r)Dv_1(\underline{x}) + \lambda_1(Dv_1(\underline{x}) - Dv_2(\underline{x})) - H_v(\underline{x}, y_1, v_1(\underline{x}), Dv_1(\underline{x}))Dv_1(\underline{x}),$$

and we can derive from (3.31) and $\rho > r$ that $\varkappa > 0$.

For $x > \underline{x}$, we write

$$Dv_1(x) = \rho(r\underline{x} + y_1)^{-\psi^{-1}}((1-\gamma)v_1(x))^{\frac{\psi^{-1}-\gamma}{1-\gamma}} + q_1(x). \tag{C.2}$$

It is clear from (C.1) that $q_1(\underline{x}) = 0$. From the continuity and boundedness of v_1 near $\underline{x}$ we know

$$((1-\gamma)v_1(x))^{\frac{\gamma-\psi^{-1}}{1-\gamma}} = ((1-\gamma)v_1(\underline{x}))^{\frac{\gamma-\psi^{-1}}{1-\gamma}} + o(1), \tag{C.3}$$

hence $q_1(x) = o(1)$ as $x \to \underline{x}$.

We notice that since $\psi^{-1} - \gamma > 0$, $((1-\gamma)v)^{\frac{\psi^{-1}-\gamma}{1-\gamma}}$ is an increasing function of v on $(-\infty, 0)$. From the concavity and monotonicity of v_1 we have

$$Dv_1(x) < Dv_1(\underline{x}) = \rho(r\underline{x} + y_1)^{-\psi^{-1}}((1-\gamma)v_1(\underline{x}))^{\frac{\psi^{-1}-\gamma}{1-\gamma}} < \rho(r\underline{x} + y_1)^{-\psi^{-1}}((1-\gamma)v_1(x))^{\frac{\psi^{-1}-\gamma}{1-\gamma}},$$

hence $q_1(x) < 0$. From $D^2v_1(x) \to -\infty$ and

$$D^2v_1(x) = \underbrace{\rho(\psi^{-1} - \gamma)(r\underline{x} + y_1)^{-\psi^{-1}}((1-\gamma)v_1(x))^{\frac{\psi^{-1}-1}{1-\gamma}}Dv_1(x)}_{>0} + Dq_1(x), \tag{C.4}$$

we have $Dq_1(x) \to -\infty$ as $x \to \underline{x}$. From (2.7) and (C.2), the consumption near $\underline{x}$ is

$$\begin{aligned}
c_1(x) &= \rho^{\psi}\left(\rho(r\underline{x} + y_1)^{-\psi^{-1}}((1-\gamma)v_1(x))^{\frac{\psi^{-1}-\gamma}{1-\gamma}} + q_1(x)\right)^{-\psi}((1-\gamma)v_1(x))^{\frac{1-\gamma\psi}{1-\gamma}} \\
&= (r\underline{x} + y_1)\left(1 + \underbrace{\rho^{-1}(r\underline{x} + y_1)^{\psi^{-1}}((1-\gamma)v_1(x))^{\frac{\gamma-\psi^{-1}}{1-\gamma}}q_1(x)}_{=o(1),\ \text{from}\ q_1(x)=o(1)\ \text{and (C.3)}}\right)^{-\psi} \\
&= r\underline{x} + y_1 - \psi\,(r\underline{x} + y_1)^{1+\psi^{-1}}((1-\gamma)v_1(x))^{\frac{\gamma-\psi^{-1}}{1-\gamma}}q_1(x) + o(q_1(x)),
\end{aligned}$$

it follows

$$s_1(x) = r(x - \underline{x}) + \psi\,(r\underline{x} + y_1)^{1+\psi^{-1}}((1-\gamma)v_1(x))^{\frac{\gamma-\psi^{-1}}{1-\gamma}}q_1(x) + o(q_1(x)).$$

Using again (C.3), we obtain

$$s_1(x) = r(x-\underline{x}) + \psi\left(r\underline{x}+y_1\right)^{1+\psi^{-1}}\left((1-\gamma)v_1(\underline{x})\right)^{\frac{\gamma-\psi^{-1}}{1-\gamma}} q_1(x) + o(q_1(x)). \tag{C.5}$$

By differentiating the HJB equation for v_1 at $x > \underline{x}$, we infer that in a right neighborhood of $\underline{x}$,

$$\begin{aligned} s_1(x)Dq_1(x) = \varkappa + &\underbrace{H_v(\underline{x}, y_1, v_1(\underline{x}), Dv_1(\underline{x}))Dv_1(\underline{x}) - H_v(x, y_1, v_1(x), Dv_1(x))Dv_1(x)}_{(I)} \\ &+ \underbrace{(\zeta - r)(Dv_1(x) - Dv_1(\underline{x})) + \lambda_1(Dv_1(x) - Dv_2(x) - Dv_1(\underline{x}) + Dv_2(\underline{x}))}_{(II)} \\ &+ \underbrace{s_1(x)\left((Dq_1(x) - D^2v_1(x)\right)}_{(III)} \\ = \varkappa + o(1). \end{aligned} \tag{C.6}$$

To obtain $(I) + (II) = o(1)$, we used the continuity of Dv_j at $\underline{x}$ (from Theorem 3.11) and the estimates on H_{vv} and H_{vp} (cf. (2.11) and (2.12)). Moreover, $(III) = o(1)$ comes from (C.4) and $s_1(x) \to 0$ as $x \to \underline{x}$. Next we denote

$$Q_1(x) := q_1^2(x), \quad Q_1(\underline{x}) = 0, \quad DQ_1(x) = 2q_1(x)Dq_1(x). \tag{C.7}$$

From (C.7), (C.5) and (C.6),

$$\underbrace{r(x-\underline{x})Dq_1(x)}_{<0} + \psi\left(r\underline{x}+y_1\right)^{1+\psi^{-1}}\left((1-\gamma)v_1(\underline{x})\right)^{\frac{\gamma-\psi^{-1}}{1-\gamma}} \frac{DQ_1(x)}{2} + o(DQ_1(x)) = \varkappa + o(1). \tag{C.8}$$

We obtain that in a right neighborhood of $\underline{x}$,

$$\psi\left(r\underline{x}+y_1\right)^{1+\psi^{-1}}\left((1-\gamma)v_1(\underline{x})\right)^{\frac{\gamma-\psi^{-1}}{1-\gamma}} \frac{DQ_1(x)}{2} > \varkappa/2.$$

This and $Q_1(0) = 0$ yield, for $x - \underline{x}$ sufficiently small,

$$Q_1(x) > \frac{\varkappa}{\psi\left(r\underline{x}+y_1\right)^{1+\psi^{-1}}\left((1-\gamma)v_1(\underline{x})\right)^{\frac{\gamma-\psi^{-1}}{1-\gamma}}}(x-\underline{x}),$$

hence $x - \underline{x} = O(Q_1(x)) = o(q_1(x))$ as $x \to \underline{x}$. This implies $r(x-\underline{x})Dq_1(x) = o(DQ_1(x))$, and then (C.8) yields

$$\frac{\psi\left(r\underline{x}+y_1\right)^{1+\psi^{-1}}\left((1-\gamma)v_1(\underline{x})\right)^{\frac{\gamma-\psi^{-1}}{1-\gamma}}}{2} DQ_1(x) + o(DQ_1(x)) = \varkappa + o(1).$$

This yields

$$Q_1(x) = \frac{2\varkappa(x-\underline{x})}{\psi\left(r\underline{x}+y_1\right)^{1+\psi^{-1}}} \cdot \left((1-\gamma)v_1(\underline{x})\right)^{\frac{\psi^{-1}-\gamma}{1-\gamma}} + o(x-\underline{x}).$$

Therefore

$$q_1(x) = -\sqrt{\frac{2\varkappa(x-\underline{x})}{\psi\left(r\underline{x}+y_1\right)^{1+\psi^{-1}}} \cdot \left((1-\gamma)v_1(\underline{x})\right)^{\frac{\psi^{-1}-\gamma}{1-\gamma}}} + o(\sqrt{x-\underline{x}}).$$

We have found that, as $x \to \underline{x}$,

$$Dv_1(x) \sim \rho(r\underline{x}+y_1)^{-\psi^{-1}}((1-\gamma)v_1(\underline{x}))^{\frac{\psi^{-1}-\gamma}{1-\gamma}} - \sqrt{\frac{2\varkappa(x-\underline{x})}{\psi\,(r\underline{x}+y_1)^{1+\psi^{-1}}}\cdot((1-\gamma)v_1(\underline{x}))^{\frac{\psi^{-1}-\gamma}{1-\gamma}}}.$$

We observe that this is consistent with the fact that Dv_1 is uniformly continuous but not Lipschitz continuous, given by Theorem 3.11 and Theorem 3.19.

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