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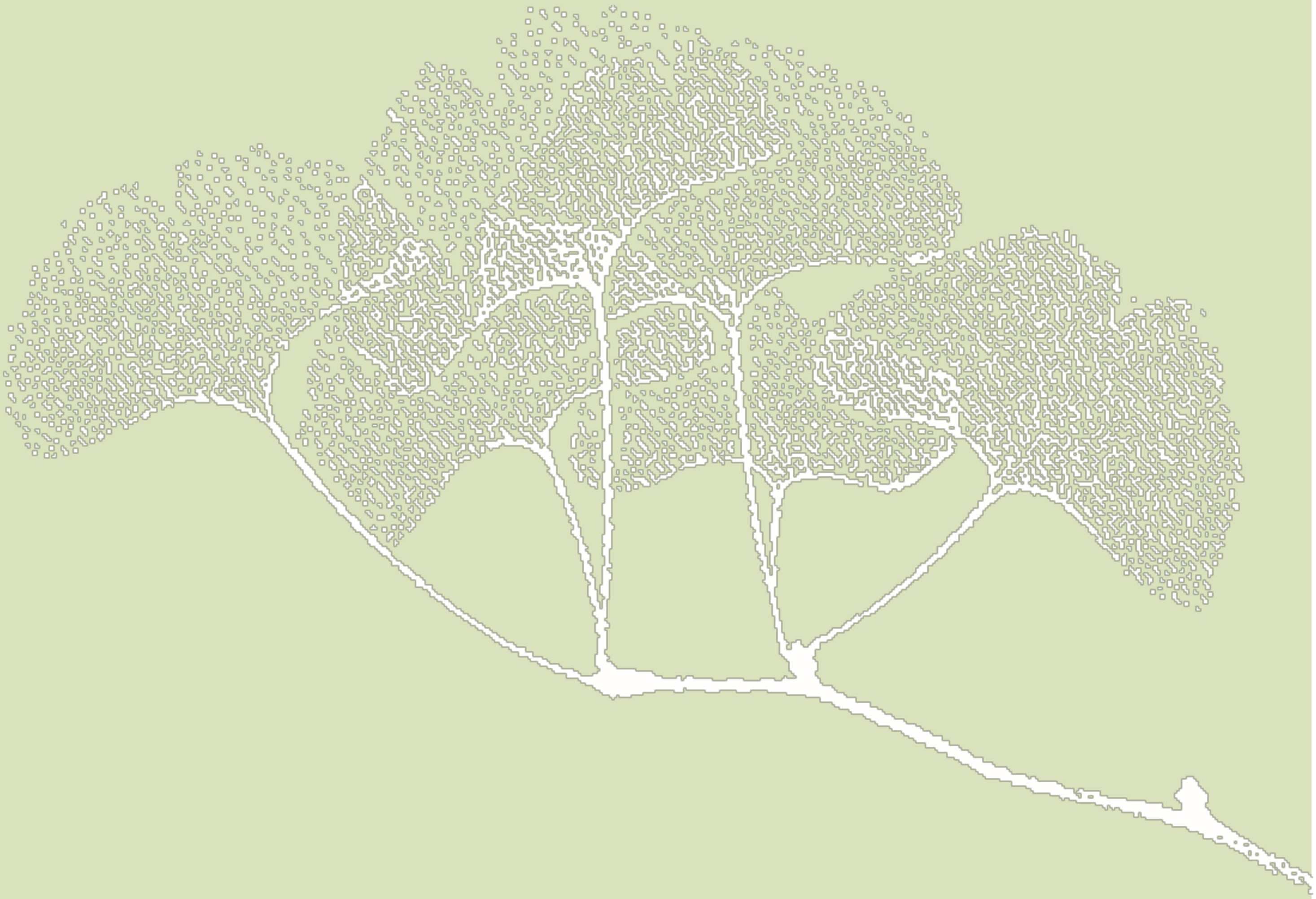

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Finite difference methods for a continuous-time heterogeneous agent model with recursive utility

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Abstract

We propose, analyze and test computational methods for solving a continuous-time heterogenous agent model with Epstein-Zin utility. Such recursive utilities allow the model to disentangle between risk aversion and intertemporal substitution. Having discretized the Hamilton-Jacobi-Bellman (HJB) equation arising in the model, we propose a Howard-Newton algorithm for the late resolution preference case, and a Howard-Tarski-Kantorovich algorithm for the early resolution preference case. We prove the convergence of the iterative algorithms. We obtain as a consequence the existence of solutions to the discretized HJB equations. In the late resolution case, we supply a priori estimates between the unique solutions of the continuous and discretized HJB equations.

Keywords: Hamilton-Jacobi-Bellman equation, Mean Field Games, recursive utility, finite difference, iterative algorithms

1 Introduction

In this work, we propose, analyze and test computational methods for solving the continuous-time heterogeneous agent (HA) models with recursive utility, recently analyzed by the authors in [4]. The continuous-time formulation of the Aiyagari-Bewley-Huggett models [17], classical in recursive macroeconomics, and the related system of partial differential equations can be studied in the light of the mathematical theory of Mean Field Games (cf. [1, 2, 15]). Such models involve a large number of *ex ante* identical but *ex post* heterogeneous agents in an incomplete market setting: each agent continuously optimizes consumption subject to a borrowing constraint and idiosyncratic income risk, resulting in a non-degenerate stationary wealth distribution (cf. [4, (1.1)]). The Mean Field Game system consists of Hamilton–Jacobi–Bellman (HJB) equations and Fokker–Planck–Kolmogorov (FPK) equations coupled through equilibrium conditions on prices. HA models have now become indispensable for quantifying the distributional effects of fiscal and monetary policy. A finite difference method for the constant relative risk aversion (CRRA) utility case was developed in [3] and its numerical appendix. This method from [3] has then become popular in the HA macro literature, as the first step for computing stationary equilibria [5, 6, 11, 12, 14]. The present paper aims at a rigorous analysis of a related finite difference method for models with recursive utility

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(Epstein-Zin). In particular, we propose convergent iterative algorithms for solving the discretized Hamilton-Jacobi-Bellman equations.

The Epstein-Zin recursive utility is defined as follows (cf. [21]):

$$f(c,v) = \frac{\rho}{1-\psi^{-1}} \frac{c^{1-\psi^{-1}} - ((1-\gamma)v)^{\theta}}{((1-\gamma)v)^{\theta-1}}, \qquad \theta = \frac{1-\psi^{-1}}{1-\gamma}, \tag{1.1}$$

where c is the consumption, v is the value function of an optimal strategy and ρ is the subjective discount rate. It is assumed that ψ, the elasticity of intertemporal substitution (EIS) and γ, the risk aversion parameter, are both positive and do not take the value 1. A key feature of the Epstein-Zin utility (1.1) is the separation between risk aversion and EIS. The time-additive separable CRRA utility is a special case of recursive utility in which $\gamma = \psi^{-1}$. The attitude towards the timing of the resolution of uncertainty is pinned down by the constant $\gamma\psi$: early (resp. late) resolution is preferred if $\gamma\psi >$ (*resp.* $<$)1, cf. [10]. With $\gamma = \psi^{-1}$, the agent is indifferent to the timing of uncertainty resolution.

The Mean Field Game system of partial differential equations for an Aiyagari model with the recursive utility (1.1) is (cf. [4, (1.7)]):

$$\begin{cases} (i) & 0 = \max_{c \geqslant 0} \{f(c, v_j) + (r^* x + y_j - c) D v_j(x)\} + \lambda_j (v_{\bar{j}}(x) - v_j(x)), \\ & c_j^*(x) = \arg\max_{c \geqslant 0} \{f(c, v_j) + (r x + y_j - c) D v_j(x)\}, \\ (ii) & -\frac{\partial}{\partial x}\left[(r^* x + y_j - c_j^*(x)) g_j(x)\right] + \lambda_{\bar{j}} g_{\bar{j}}(x) - \lambda_j g_j(x) = 0, \\ & \int_{x \geqslant \underline{x}} g_1(x) dx + \int_{x \geqslant \underline{x}} g_2(x) dx + \sum_{j \in \{1,2\}} \mu_j = 1, \\ (iii_A) & r^* = A\alpha \left(\frac{\mathcal{K}[r^*]}{N}\right)^{\alpha-1} - \delta = A\alpha \left(\frac{K[m^{(r^*)}]}{N}\right)^{\alpha-1} - \delta. \end{cases} \tag{1.2}$$

The HJB equations (1.2) (i), one for each j ($j \in \{1,2\}$ and $\bar{j} = 3 - j$) are coupled through the switching terms $\lambda_j(v_{\bar{j}} - v_j)$, encoding the transitions between income states with Poisson rates λ_j. The state constraint $x \geqslant \underline{x}$ (a borrowing limit) is imposed to prevent negative savings at $\underline{x}$: $r^*\underline{x} + y_j - c_j^*(\underline{x}) \geqslant 0$. Each of the stationary FPK equations (1.2) (ii) describes the density g_j of the invariant measure m_j associated with the agents with income y_j. The measure m_j possibly exhibits a Dirac mass at $\underline{x}$, weighted by μ_j. The equilibrium interest rate r^* is determined by the fixed-point condition (1.2) (iii_A), where the aggregate capital stock $\mathcal{K}[r^*] = K[m^{(r^*)}] = \sum_j \int_{x \geqslant \underline{x}} x dm_j^{(r^*)}$ is itself a functional of the invariant measure (which depends on r^*), and thus closes the model. Although the *Mean Field* coupling (1.2) (iii_A) is particular, the analysis developed below applies to a wide range of HA models.

To solve the state constrained HJB equation (i) with a given $r > 0$, we use monotone (upwind) finite difference schemes, as in [1, 3, 8, 16, 20]. The main difficulties in this model are: firstly, the Hamiltonian takes infinite values for negative momentum; secondly (this is specific to the recursive utility) the dependence of f on v. The second difficulty implies that one should use iterative algorithms that differs from the standard Howard algorithms used for example in [3, 8]. A main discovery of the present paper is that different strategies are suitable depending on the timing preference in the models. In the late resolution case $\theta \geqslant 1$, the system satisfies a comparison principle, and we use a Howard-Newton algorithm to solve for the unique solution. In this method, the policy evaluation step consists of an inner loop of Newton iterations. In the early resolution case $0 < \theta < 1$, we consider a solution to the HJB equation as a fixed point of a suitably defined map Γ, and take advantage of the monotonicity and invariance properties of Γ to design a Howard-Tarski-Kantorovich algorithm. We have decided to name the method so because the design of the outer loop is reminiscent of Tarski-Kantorovich fixed point theorem and its proof in [13, Theorem I, p.68]. We prove the convergence of these iterative algorithms, and this also implies the existence of solutions to the discretized HJB equations. The Fokker-Planck-Kolmogorov equations will then be discretized with a finite difference scheme, exploiting the duality structure of (1.2) (cf. [3]).

The methodology for studying the case $0 < \theta < 1$ in the present paper is related to the recent literature about dynamic programming on ordered spaces and Koopman operators, e.g. [7, 18, 19]. In these works, the Tarski-Kantorovich fixed point theorem have been used to study discrete time Bellman equations with recursive utility. In the present paper, we show that the barriers and invariance properties, essential for using this method, can be constructed for the upwind schemes that we propose.

We analyze the convergence of the numerical solution to the constrained viscosity solution of the HJB equation, in the case of $\theta \geqslant 1$. Recently, a Semi-Lagrangian method was proposed in [9] to study continuous-time HA models with CRRA utility, with convergence analysis based on the Barles-Souganidis half-relaxed limits method. In the present paper, we obtain error estimates with a technique that does not require doubling the variables, thanks to the strong regularity of the solutions to (1.2) (i).

2 Preliminaries

The assumptions that follow will be made in the whole paper:

$$\gamma > 1,\ 0 < \psi < 1, \quad \rho > r > 0, \quad y_2 > y_1 > 0, \quad \rho\underline{x} + y_1 > 0. \tag{2.1}$$

It has been proved in [4] that the equilibrium interest rate r^* is smaller than ρ, if there exists a solution to (1.2). This justifies the bounds on r in (2.1). With the constant b defined in Table 1 below, (2.1) yields $r < b < \rho$, $rx + y_2 > rx + y_1 > 0$ and $b(x + y_1/r) > 0$. Therefore, the functions in (2.11) below are well defined.

With Epstein-Zin utility (1.1), we can rewrite the HJB equation (1.2) (i)

$$\frac{\rho}{\theta} v_j(x) = H(x, y_j, v_j, Dv_j) + \lambda_j(v_{\bar{j}}(x) - v_j(x)), \tag{2.2}$$

where we use the notation

$$f(c, v_j) = \mathcal{F}(c, v_j) - \frac{\rho}{\theta} v_j, \qquad \mathcal{F}(c, v_j) = \frac{\rho}{1-\psi^{-1}} \frac{c^{1-\psi^{-1}}}{((1-\gamma)v_j)^{\theta-1}}, \tag{2.3}$$

$$H(x, y_j, v_j, p) := \max_{c \geqslant 0} \left\{ \mathcal{F}(c, v_j) + (rx + y_j - c)p \right\}. \tag{2.4}$$

From the first order necessary optimality condition, the Hamiltonian in (2.4) is

$$H(x, y, v, p) = \begin{cases} (rx + y)p + \dfrac{\rho^{\psi}}{\psi - 1} p^{1-\psi} ((1-\gamma)v)^{\frac{1-\gamma\psi}{1-\gamma}}, & \text{if} \quad p \geqslant 0, \\ +\infty, & \text{if} \quad p < 0. \end{cases} \tag{2.5}$$

We observe that $H(x,y,v,p)$ defined in (2.5) is strictly convex in p for fixed (x,y,v) with $p>0$, and that

$$\min_{p>0} H(x,y,v,p) = \rho\frac{(rx+y)^{1-\psi^{-1}}}{1-\psi^{-1}}\big((1-\gamma)v\big)^{\frac{\psi^{-1}-\gamma}{1-\gamma}}. \tag{2.6}$$

Moreover, since $rx+y>0$ and $p^{1-\psi}$ is sublinear, we infer from (2.5) the coercivity property

$$H(x,y,v,p)\to+\infty \quad \text{when} \quad p\to+\infty. \tag{2.7}$$

The optimal consumption (away from the borrowing limit) is given by the first order necessary optimality condition, whenever $Dv_j>0$,

$$c_j = \arg\max_{c\geqslant 0}\{\mathcal{F}(c,v_j)-cDv_j\} = \rho^{\psi}(Dv_j)^{-\psi}((1-\gamma)v_j)^{\frac{1-\gamma\psi}{1-\gamma}}. \tag{2.8}$$

We summarize the notation in Table 1.

Table 1: Symbols

Subjective discount factor	$\rho>0$
Risk aversion	$\gamma>1$
Elasticity of intertemporal substitution (EIS)	$0<\psi<1$
A parameter arising in Epstein-Zin utility	$\theta:=\frac{1-\psi^{-1}}{1-\gamma}>0$
A convenient parameter	$b:=\rho\left[\frac{r+\psi(\rho-r)}{\rho}\right]^{\frac{1}{1-\psi}}$
Aggregator	$f(c,v)$
Modified aggregator	$\mathcal{F}(c,v)=f(c,v)+\frac{\rho}{\theta}v$
Borrowing limit	$\underline{x}$

For any $c>0$, if $\theta>1$ (resp. $0<\theta<1$) then $\mathcal{F}(c,v)$ is decreasing (resp. increasing) in v, i.e.

$$\mathcal{F}_v(c,v)<0 \quad (\textit{resp.}\ \mathcal{F}_v(c,v)>0). \tag{2.9}$$

If $\theta>1$, then $f(c,v)$ and $\mathcal{F}(c,v)$ are jointly concave in $(c,v)\in(0,+\infty)\times(-\infty,0)$.

We denote by $H_v(x,y,v,p)$ and $H_{vv}(x,y,v,p)$ the first and second order derivatives of $H(x,y,v,p)$ with respect to v. We use $H_{vp}(x,y,v,p)$ to denote the cross second order derivative of $H(x,y,v,p)$ with respect to v and p. Straightforward computations lead to:

$$\begin{aligned}&H_v(x,y,v,p)<0,\ H_{vp}(x,y,v,p)<0,\ H_{vv}(x,y,v,p)<0 \text{ if } \theta>1,\\ &H_v(x,y,v,p)>0,\ H_{vp}(x,y,v,p)>0,\ H_{vv}(x,y,v,p)>0 \text{ if } 0<\theta<1.\end{aligned} \tag{2.10}$$

Therefore, $H(x,y,v,p)$ is strictly concave (resp. convex) in the v variable if $\theta>1$ (resp. $0<\theta<1$).

Definition 2.1.

1. *A continuous function $v=(v_1,v_2)$ is said to be a viscosity subsolution of (2.2) at x, if whenever φ is a smooth function and $v_j-\varphi$ has a local maximum at x, then*

$$\frac{\rho}{\theta}v_j(x)\leqslant H(x,y_j,v_j(x),D\varphi(x))+\lambda_j(v_{\bar{j}}(x)-v_j(x)).$$

2. A continuous function $v = (v_1, v_2)$ is said to be a viscosity supersolution of (2.2) at x, if whenever φ is a smooth function and $v_j - \varphi$ has a local minimum at x, then

$$\frac{\rho}{\theta} v_j(x) \geqslant H(x, y_j, v_j(x), D\varphi(x)) + \lambda_j(v_{\bar{j}}(x) - v_j(x)).$$

3. A continuous function v is said to be a constrained viscosity solution to system (2.2) if v is a viscosity supersolution in $(\underline{x}, \infty)$ and a viscosity subsolution in $[\underline{x}, \infty)$.

The sub- and supersolutions proposed below will play an important role in what follows. The following result has been proved in [4, Proposition 3.5] for the case $\theta \geqslant 1$. We observe that the same proof holds for $0 < \theta < 1$.

Proposition 2.2. *With the constant b defined in Table 1, we consider the functions*

$$\check{\mathsf{u}}_1(x) = \check{\mathsf{u}}_2(x) = \frac{(rx + y_1)^{1-\gamma}}{1-\gamma}, \quad \check{\mathsf{v}}_1(x) = \check{\mathsf{v}}_2(x) = \frac{(b(x + y_2/r))^{1-\gamma}}{1-\gamma}. \tag{2.11}$$

The pairs $(\check{\mathsf{u}}_1, \check{\mathsf{u}}_2)$ and $(\check{\mathsf{v}}_1, \check{\mathsf{v}}_2)$ are respectively a subsolution of (2.2) in $[\underline{x}, +\infty)$ and a supersolution of (2.2) in $(\underline{x}, +\infty)$.

We next state a comparison principle for viscosity sub- and supersolutions of (2.2), in the late resolution case. It follows directly from [4, Proposition 3.3].

Proposition 2.3. *Assume $\theta \geqslant 1$, $\mathsf{u} = (\mathsf{u}_1, \mathsf{u}_2)$ and $\mathsf{v} = (\mathsf{v}_1, \mathsf{v}_2)$ are bounded viscosity sub- and supersolution of system (2.2). We extend $\mathbf{v}_j$ at $\underline{x}$ by setting $\mathsf{v}_j(\underline{x}) = \lim_{z \to \underline{x}, z > \underline{x}} \mathsf{v}_j(z)$. Then $\mathsf{u} \leqslant \mathsf{v}$ in $[\underline{x}, +\infty)$, i.e. $\mathsf{u}_j \leqslant \mathsf{v}_j$ in $[\underline{x}, +\infty)$ for $j = 1, 2$.*

Proposition 2.3 yields the uniqueness of the viscosity solution when $\theta \geqslant 1$. Reference [4] contains a complete study of continuous-time HA models in the case $\theta \geqslant 1$. It deals in particular with the existence, uniqueness and regularity of solution to the HJB equation. The following proposition summarizes results from [4, Proposition 3.7, Proposition 3.9, Proposition 3.10, Proposition 3.15, Proposition 3.16] .

Proposition 2.4. *Assume $\theta \geqslant 1$. There exists a unique viscosity solution (v_1, v_2) which is C^1 and strictly concave. Moreover, $v_j \in W^{2,\infty}_{loc}(\underline{x}, +\infty)$ and $v_2 \geqslant v_1$.*

Next we give some results concerning the saving policies s_j. The following proposition summarizes results from [4, Proposition 3.17, Corollary 3.18, Proposition 3.21].

Proposition 2.5. *Assume $\theta \geqslant 1$. The optimal saving policy s_1 has the following properties: $s_1(x) < 0$ for all $x > \underline{x}$ and $s_1(\underline{x}) = 0$. If furthermore*

$$(\rho - r)\,(r\underline{x} + y_2)^{-1/\psi} + \lambda_2 \left((r\underline{x} + y_2)^{-1/\psi} - (r\underline{x} + y_1)^{-1/\psi} \right) < 0, \tag{2.12}$$

then $s_2(\underline{x}) > 0$. Moreover, $Dv_1(\underline{x}) > Dv_2(\underline{x})$.

Remark 2.6. *In the analysis of the numerical scheme for $\theta \geqslant 1$, we will focus on situations in which $s_2(\underline{x}) > 0$, because at the mean field equilibrium described by (1.2), the state constraint is not binding for agents with high income. Nevertheless, note that the methods described below apply even if (2.12) is not satisfied. It has been shown in [4] that in this case $s_2(\underline{x}) = 0$ and $s_2 < 0$ in some interval $(\underline{x}, \underline{x} + \epsilon)$. If furthermore, $(\rho/\theta - r)\,(r\underline{x} + y_2)^{-1/\psi} + \lambda_2 \left((r\underline{x} + y_2)^{-1/\psi} - (r\underline{x} + y_1)^{-1/\psi} \right) \geqslant 0$, then $s_2(x) < 0$ for all $x > \underline{x}$. We refer to Section 6 below for an example.*

The next result deals with the behavior of s_2 as $x \to +\infty$. It justifies solving the HJB equation numerically on a bounded domain $[\underline{x}, \bar{x}]$.

Proposition 2.7 ([4], Proposition 3.22)**.** *Assume $\theta \geqslant 1$ and $s_2(\underline{x}) > 0$. There exists $\widehat{x} > \underline{x}$ such that $s_2(\widehat{x}) = 0$ and $s_2(x) < 0$ for all $x \geqslant \widehat{x}$.*

Remark 2.8. *Proposition 2.7 implies that if $\bar{x} > \widehat{x}$, then imposing an artificial state constraint at $\bar{x}$ while solving* (2.2) *does not change the solution.*

It has been shown in [4, Corollary 3.19] that, under the same assumptions as Proposition 2.5, $\lim_{x\to\underline{x}} D^2 v_1(x) = -\infty$. This will require special care in proving the convergence of the finite difference scheme below. The next result, from [4, Appendix C], is about the asymptotic analysis of s_1, $D^2 v_1$ and Dv_1 near $x = \underline{x}$.

Proposition 2.9. *We make the same assumptions as in Proposition 2.5. Near $x = \underline{x}$, the functions s_1, $D^2 v_1$ and Dv_1 have the following behavior:*

$$s_1(x) D^2 v_1(x) = \varkappa + o(1), \tag{2.13}$$

$$Dv_1(x) \sim \rho \frac{((1-\gamma)v_1(\underline{x}))^{\frac{\psi^{-1}-\gamma}{1-\gamma}}}{(r\underline{x}+y_1)^{\psi^{-1}}} - \sqrt{\frac{2\varkappa(x-\underline{x})}{\psi\,(r\underline{x}+y_1)^{1+\psi^{-1}}}((1-\gamma)v_1(\underline{x}))^{\frac{\psi^{-1}-\gamma}{1-\gamma}}}, \tag{2.14}$$

where $\varkappa$ is given by

$$\varkappa := \left(\frac{\rho}{\theta} - r\right) Dv_1(\underline{x}) + \lambda_1(Dv_1(\underline{x}) - Dv_2(\underline{x})) - H_v(\underline{x}, y_1, v_1(\underline{x}), Dv_1(\underline{x}))Dv_1(\underline{x}). \tag{2.15}$$

Lemma 2.10. *We make the same assumptions as in Proposition 2.5. There exists $\eta \in (\underline{x}, \bar{x})$ such that for all $x \in (\underline{x}, \eta)$,*

$$0 < -Dv_1^2(x) \leqslant \sqrt{\frac{\varkappa}{\psi\,(r\underline{x}+y_1)^{1+\psi^{-1}}(x-\underline{x})} \cdot ((1-\gamma)v_1(\underline{x}))^{\frac{\psi^{-1}-\gamma}{1-\gamma}}}. \tag{2.16}$$

and

$$0 < -s_1(x) < 2\sqrt{\varkappa\psi\,(r\underline{x}+y_1)^{1+\psi^{-1}}((1-\gamma)v_1(\underline{x}))^{\frac{\psi^{-1}-\gamma}{\gamma-1}}(x-\underline{x})} \tag{2.17}$$

Proof. From [4, Appendix C], we know $Dv_1^2(x) \sim Dq_1(x)$, where $q_1(x)$ and $Q_1(x) = q_1^2(x)$ satisfy

$$q_1(x) = -\sqrt{\frac{2\varkappa(x-\underline{x})}{\psi\,(r\underline{x}+y_1)^{1+\psi^{-1}}} \cdot ((1-\gamma)v_1(\underline{x}))^{\frac{\psi^{-1}-\gamma}{1-\gamma}}} + o(\sqrt{x-\underline{x}}),$$

$$\frac{\psi\,(r\underline{x}+y_1)^{1+\psi^{-1}}((1-\gamma)v_1(\underline{x}))^{\frac{\gamma-\psi^{-1}}{1-\gamma}}}{2} DQ_1(x) + o(DQ_1(x)) = \varkappa + o(1).$$

(2.16) follows from $Dq_1(x) = \frac{DQ_1(x)}{2q(x)}$. From Proposition 2.5 we know $-s_1(x) > 0$ for all $x > \underline{x}$, then (2.17) follows from (2.16) and (2.13). ◻

Proposition 2.9, Lemma 2.10 and the fact that $v_j \in W^{2,\infty}_{loc}(\underline{x}, \bar{x}]$ will be used in Section 4.3 below to obtain an explicit convergence rate.

The analysis of (1.2) (i) is therefore rather complete in the case $\theta \geqslant 1$. On the contrary, many things remain to be done in the case $0 < \theta < 1$. In the present paper, we restrict ourselves to supplying some results that are useful in the study of the numerical scheme.

Let us introduce the following system, for a given pair of functions $(\widetilde{v}_1, \widetilde{v}_2)$:

$$\begin{cases} \dfrac{\rho}{\theta} v_j(x) = H(x, y_j, \widetilde{v}_j, Dv_j) + \lambda_j(v_{\bar{j}}(x) - v_j(x)), \\ \dfrac{\rho}{\theta} v_j(x) = H(x, y_j, \widetilde{v}_j, Dv_j) + \lambda_j(v_{\bar{j}}(x) - v_j(x)). \end{cases} \tag{2.18}$$

Proposition 2.11. *Assume $\widetilde{v}_j$ is locally Lipschitz and $\check{\mathsf{u}}_j \leqslant \widetilde{v}_j \leqslant \check{\mathsf{v}}_j$. Let $\mathsf{u} = (\mathsf{u}_1, \mathsf{u}_2)$ and $\mathsf{v} = (\mathsf{v}_1, \mathsf{v}_2)$ be bounded viscosity sub- and supersolution of system* (2.18)*. Then $\mathsf{u} \leqslant \mathsf{v}$ in $[\underline{x}, +\infty)$.*

Proposition 2.12. *Assume $\widetilde{v}_j$ is locally Lipschitz and $\check{\mathsf{u}}_j \leqslant \widetilde{v}_j \leqslant \check{\mathsf{v}}_j$. There exists a unique viscosity solution (v_1, v_2) to* (2.18)*. Moreover, the solution $v_j \in C^1[\underline{x}, +\infty)$.*

We define the map Γ: $(v_1, v_2) = \Gamma(\widetilde{v}_1, \widetilde{v}_2)$ if and only if (v_1, v_2) is the unique viscosity solution of (2.18). A solution to (2.2) can then be defined as a fixed point of Γ.

The proposition that follows states that the map Γ is monotone when $0 < \theta < 1$:

Proposition 2.13. *Assume $0 < \theta < 1$. Let $(v_1, v_2) = \Gamma(\widetilde{v}_1, \widetilde{v}_2)$ and $(u_1, u_2) = \Gamma(\widetilde{u}_1, \widetilde{u}_2)$. If $\widetilde{u}_j \leqslant \widetilde{v}_j$, then $u_j \leqslant v_j$.*

Proof. Since $\widetilde{u}_j \leqslant \widetilde{v}_j$, we deduce $H(x, y_j, \widetilde{u}_j, Du_j) \leqslant H(x, y_j, \widetilde{v}_j, Du_j)$ from (2.10). This implies (u_1, u_2) is a subsolution of the system of HJB equations (2.18) satisfied by (v_1, v_2). The result then follows from Proposition 2.11. □

Proposition 2.13 yields the following invariance principle.

Proposition 2.14. *We use the same notation as in Proposition 2.13. If $0 < \theta < 1$ and $\check{\mathsf{u}}_j \leqslant \widetilde{v}_j \leqslant \check{\mathsf{v}}_j$, then $\check{\mathsf{u}}_j \leqslant v_j \leqslant \check{\mathsf{v}}_j$.*

From Proposition 2.13 and Proposition 2.14, with $0 < \theta < 1$ Tarski's fixed point theorem implies that there exists a fixed point of Γ. Since Proposition 2.3 may not hold with $0 < \theta < 1$, the uniqueness of a constrained viscosity solution to (2.2) remains an open question.

Proposition 2.15. *For $0 < \theta < 1$, there exists a viscosity solution (v_1, v_2) of* (2.2) *that satisfies $\check{\mathsf{u}}_j \leqslant v_j \leqslant \check{\mathsf{v}}_j$.*

Oberve that Proposition 2.3 may not hold if $0 < \theta < 1$, while the invariance principle in Proposition 2.14 may not hold if $\theta \geqslant 1$. This is a reason why different algorithms will be needed in the two intertemporal preference cases.

3 The finite difference method

3.1 The numerical schemes

Given $\underline{x} \leqslant 0 < \bar{x}$, $I \in \mathbb{N}$ and a step size $\Delta x = (\bar{x} - \underline{x})/I$, let us define the grid $\mathcal{G}^{\Delta x} = \{x_i : i \in \mathbb{N},\ i = 0, \cdots, I,\ x_i = \underline{x} + i\Delta x\}$. For two grid functions U_j and V_j defined on $\mathcal{G}^{\Delta x}$, we use $U_j \preceq V_j$

(resp. $U_j \succeq V_j$) to denote the ordering $U_{i,j} \leqslant V_{i,j}$ (resp. $U_{i,j} \geqslant V_{i,j}$) for all i. If $U = (U_1, U_2)$ and $V = (V_1, V_2)$, then $U \preceq V$ (resp. $U \succeq V$) means $U_{i,j} \leqslant V_{i,j}$ (resp. $U_{i,j} \geqslant V_{i,j}$) for all i, j.

For a Hamiltonian strictly convex w.r.t. the $p-$variable, we set

$$\begin{aligned} H_{\min}\left(x_i, y_j, V_{i,j}\right) &= \min_p H\left(x_i, y_j, V_{i,j}, p\right), \\ \text{and} \quad p_{\min}\left(x_i, y_j, V_{i,j}\right) &= \arg\min_p H\left(x_i, y_j, V_{i,j}, p\right). \end{aligned} \tag{3.1}$$

The discrete Hamiltonian is defined by

$$\begin{aligned} &\boldsymbol{H}\left(x_i, y_j, V_{i,j}, \boldsymbol{p}^F, \boldsymbol{p}^B\right) \\ &= \begin{cases} H^{\uparrow}\left(x_i, y_j, V_{i,j}, \boldsymbol{p}^F\right) & \text{if } x_i = \underline{x}, \\ H^{\uparrow}\left(x_i, y_j, V_{i,j}, \boldsymbol{p}^F\right) + H^{\downarrow}\left(x_i, y_j, V_{i,j}, \boldsymbol{p}^B\right) - H_{\min}\left(x_i, y_j, V_{i,j}\right) & \text{if } \underline{x} < x_i < \bar{x}, \\ H^{\downarrow}\left(x_i, y_j, V_{i,j}, \boldsymbol{p}^B\right) & \text{if } x_i = \bar{x}. \end{cases} \end{aligned} \tag{3.2}$$

where

$$H^{\uparrow}\left(x_i, y_j, V_{i,j}, \boldsymbol{p}^F\right) = \begin{cases} H\left(x_i, y_j, V_{i,j}, \boldsymbol{p}^F\right) & \text{if} \quad \boldsymbol{p}^F \geqslant p_{\min}\left(x_i, y_j, V_{i,j}\right), \\ H_{\min}\left(x_i, y_j, V_{i,j}\right) & \text{if} \quad \boldsymbol{p}^F < p_{\min}\left(x_i, y_j, V_{i,j}\right), \end{cases} \tag{3.3}$$

and

$$H^{\downarrow}\left(x_i, y_j, V_{i,j}, \boldsymbol{p}^B\right) = \begin{cases} H\left(x_i, y_j, V_{i,j}, \boldsymbol{p}^B\right) & \text{if} \quad 0 \leqslant \boldsymbol{p}^B \leqslant p_{\min}\left(x_i, y_j, V_{i,j}\right), \\ H_{\min}\left(x_i, y_j, V_{i,j}\right) & \text{if} \quad \boldsymbol{p}^B > p_{\min}\left(x_i, y_j, V_{i,j}\right), \\ +\infty & \text{if} \quad \boldsymbol{p}^B < 0. \end{cases} \tag{3.4}$$

Lemma 3.1. *The numerical Hamiltonian $\boldsymbol{H}\left(x_i, y_j, V_{i,j}, \boldsymbol{p}^F, \boldsymbol{p}^B\right)$ is increasing in $\boldsymbol{p}^F$ and decreasing in $\boldsymbol{p}^B$. Moreover,*

$$H^{\downarrow}\left(x_i, y_j, V_{i,j}, \boldsymbol{p}^B\right) \leqslant 0 \quad \textit{if} \quad \boldsymbol{p}^B \geqslant 0. \tag{3.5}$$

Proof. From the strict convexity of $H\left(x_i, y_j, V_{i,j}, p\right)$ w.r.t. p and (3.1), we deduce for each $\left(x_i, y_j, V_{i,j}\right)$, $H\left(x_i, y_j, V_{i,j}, \cdot\right)$ is increasing in the domain $[p_{\min}, +\infty)$ and decreasing in $[0, p_{\min})$. □

In the design of Howard type algorithms, it will be crucial to take advantage of the following relationship between the discrete Hamiltonian and the utility function (2.4):

$$s_{i,j}^F = rx_i + y_j - c_{i,j}^F, \quad s_{i,j}^B = rx_i + y_j - c_{i,j}^B, \tag{3.6}$$

$$\begin{aligned} H^{\uparrow}\left(x_i, y_j, V_{i,j}, \boldsymbol{p}^F\right) &= \sup_{c_{i,j}^F \in [0, rx_i + y_j]} \left\{ s_{i,j}^F \boldsymbol{p}^F + \mathcal{F}\left(c_{i,j}^F, V_{i,j}\right) \right\}, \\ H^{\downarrow}\left(x_i, y_j, V_{i,j}, \boldsymbol{p}^B\right) &= \sup_{c_{i,j}^B \geqslant rx_i + y_j} \left\{ s_{i,j}^B \boldsymbol{p}^B + \mathcal{F}\left(c_{i,j}^B, V_{i,j}\right) \right\}. \end{aligned} \tag{3.7}$$

Next, we denote the consumption associated to null savings by:

$$\bar{c}_j(x_i) = rx_i + y_j. \tag{3.8}$$

A straightforward computation yields

$$H_{\min}\left(x_i, y_j, V_{i,j}\right) = \mathcal{F}\left(\bar{c}_j(x_i), V_{i,j}\right).$$

We denote the forward and backward finite difference operators by

$$\Delta^+ V_{i,j} = \frac{V_{i+1,j} - V_{i,j}}{\Delta x}, \ \forall\ 0 \leqslant i < I \text{ and } \Delta^- V_{i,j} = \frac{V_{i,j} - V_{i-1,j}}{\Delta x}, \ \forall\ 0 < i \leqslant I. \tag{3.9}$$

For a function v defined on $[\underline{x}, +\infty)$, we denote for $x \in \mathcal{G}^{\Delta x}$,

$$\begin{aligned} \Delta^+ v(x) &= \frac{v(x+\Delta x) - v(x)}{\Delta x}, \text{ if } \underline{x} < \bar{x}, \\ \Delta^- v(x) &= \frac{v(x) - v(x-\Delta x)}{\Delta x}, \text{ if } \underline{x} < x \leqslant \bar{x}. \end{aligned} \tag{3.10}$$

We can set $\Delta^+ V_{I,j}$, $\Delta^- V_{0,j}$, $\Delta^+ v(\bar{x})$, $\Delta^- v(\underline{x})$ to arbitrary constants since their values have no importance in what follows..

The discrete version of the HJB equation (2.2) is, for $i = 0, \cdots, I$,

$$\begin{cases} \dfrac{\rho}{\theta} V_{i,1} = \boldsymbol{H}\left(x_i, y_1, V_{i,1}, \Delta^+ V_{i,1}, \Delta^- V_{i,1}\right) + \lambda_1 \left(V_{i,2} - V_{i,1}\right), \\ \dfrac{\rho}{\theta} V_{i,2} = \boldsymbol{H}\left(x_i, y_2, V_{i,2}, \Delta^+ V_{i,2}, \Delta^- V_{i,2}\right) + \lambda_2 \left(V_{i,1} - V_{i,2}\right). \end{cases} \tag{3.11}$$

Observe that by design of $\boldsymbol{H}$ in (3.2) at $x_i = \underline{x}$ and $x_i = \bar{x}$, the equation (3.11) at $i = 0$ and $i = I$ have taken into account the state constraints because $H^{\uparrow}$ (resp. $H^{\downarrow}$) corresponds to nonnegative (nonpositive) savings. If $\Delta^- V_{i,j} > 0$ for all $1 \leqslant i \leqslant I$ (this will be proved in Proposition 4.4 below), then from the first order condition of optimality, we deduce the numerical optimal consumption

$$\begin{cases} c_{i,j}^{F,*} = \min\left\{\rho^{\psi}\left(\Delta^+ V_{i,j}\right)^{-\psi}\left((1-\gamma)V_{i,j}\right)^{\frac{1-\gamma\psi}{1-\gamma}}, \bar{c}_j(x_i)\right\} \quad \text{for } x_i < \bar{x}, \\ c_{i,j}^{B,*} = \max\left\{\rho^{\psi}\left(\Delta^- V_{i,j}\right)^{-\psi}\left((1-\gamma)V_{i,j}\right)^{\frac{1-\gamma\psi}{1-\gamma}}, \bar{c}_j(x_i)\right\} \quad \text{for } x_i > \underline{x}, \\ c_{0,j}^{B,*} = r\underline{x} + y_j, \quad c_{I,j}^{F,*} = r\bar{x} + y_j. \end{cases} \tag{3.12}$$

An equivalent formulation of (3.11) is then:

$$\begin{cases} \dfrac{\rho}{\theta} V_{i,1} = s_{i,1}^{F,*} \Delta^+ V_{i,1} + s_{i,1}^{B,*} \Delta^- V_{i,1} + \mathcal{F}\left(c_{i,1}^{F,*}, V_{i,1}\right) + \mathcal{F}\left(c_{i,1}^{B,*}, V_{i,1}\right) \\ \qquad\qquad - \mathcal{F}\left(\bar{c}_{i,1}, V_{i,1}\right) + \lambda_1 \left(V_{i,2} - V_{i,1}\right), \\ \dfrac{\rho}{\theta} V_{i,2} = s_{i,2}^{F,*} \Delta^+ V_{i,2} + s_{i,2}^{B,*} \Delta^- V_{i,2} + \mathcal{F}\left(c_{i,1}^{F,*}, V_{i,2}\right) + \mathcal{F}\left(c_{i,1}^{B,*}, V_{i,2}\right) \\ \qquad\qquad - \mathcal{F}\left(\bar{c}_{i,2}, V_{i,2}\right) + \lambda_2 \left(V_{i,1} - V_{i,2}\right), \end{cases} \tag{3.13}$$

where $\mathcal{F}$ is defined as in (2.3), $s_{i,j}^{F,*} = rx_i + y_j - c_{i,j}^{F,*}$ and $s_{i,j}^{B,*} = rx_i + y_j - c_{i,j}^{B,*}$. By construction $s_{i,j}^{F,*} \geqslant 0$ and $s_{i,j}^{B,*} \leqslant 0$, hence the scheme is in upwind form.

With $\bar{c}_j$ defined in (3.8), and given $\varepsilon > 0$, let us introduce the following regularized consumption policies:

$$\begin{cases} c_{i,j}^{F,*} = \min\left\{\rho^{\psi}\left(\Delta^+ V_{i,j}\right)_+^{-\psi}\left((1-\gamma)V_{i,j}\right)^{\frac{1-\gamma\psi}{1-\gamma}}, \bar{c}_j(x_i)\right\} \quad \text{for } x_i < \bar{x}, \\ c_{i,j}^{B,*} = \max\left\{\min\left\{1/\varepsilon, \rho^{\psi}\left(\Delta^- V_{i,j}\right)_+^{-\psi}\left((1-\gamma)V_{i,j}\right)^{\frac{1-\gamma\psi}{1-\gamma}}\right\}, \bar{c}_j(x_i)\right\} \quad \text{for } x_i > \underline{x}, \\ c_{0,j}^{B,*} = r\underline{x} + y_j, \quad c_{I,j}^{F,*} = r\bar{x} + y_j. \end{cases} \tag{3.14}$$

Let us introduce the regularized Hamiltonian, with $\left(c^{F,*}, c^{B,*}\right)$ and V satisfying (3.14):

$$\begin{aligned}&\boldsymbol{H}_\varepsilon\left(x_i, y_j, V_{i,j}, \boldsymbol{p}^F, \boldsymbol{p}^B\right)\\&= s_{i,j}^{F,*}\boldsymbol{p}^F, + s_{i,j}^{B,*}\boldsymbol{p}^B + \mathcal{F}\left(c_{i,j}^{F,*}, V_{i,j}\right) + \mathcal{F}\left(c_{i,j}^{B,*}, V_{i,j}\right) - \mathcal{F}\left(\bar{c}_j(x_i), V_{i,j}\right).\end{aligned} \tag{3.15}$$

We introduce a regularized version of (3.11):

$$\left\{\begin{aligned}\frac{\rho}{\theta}V_{i,1} &= \boldsymbol{H}_\varepsilon\left(x_i, y_1, V_{i,1}, \Delta^+V_{i,1}, \Delta^-V_{i,1}\right) + \lambda_1\left(V_{i,2} - V_{i,1}\right),\\ \frac{\rho}{\theta}V_{i,2} &= \boldsymbol{H}_\varepsilon\left(x_i, y_2, V_{i,2}, \Delta^+V_{i,2}, \Delta^-V_{i,2}\right) + \lambda_2\left(V_{i,1} - V_{i,2}\right),\end{aligned}\right. \tag{3.16}$$

supposing $((1-\gamma)V_{i,j})^{\frac{1-\gamma\psi}{1-\gamma}}$ is bounded (we shall see later that this property is true, uniformly with respect to ε). Throughout the rest of the present paper, we assume

$$1/\varepsilon > r\bar{x} + y_2. \tag{3.17}$$

We observe that the regularized Hamiltonian in (3.16) is defined even if $\Delta^- V_{i,j} < 0$. We also set

$$\begin{aligned}H_\varepsilon^\uparrow\left(x_i, y_j, V_{i,j}, \Delta^+V_{i,j}\right) &= H^\uparrow\left(x_i, y_j, V_{i,j}, \Delta^+V_{i,j}\right),\\ H_\varepsilon^\downarrow\left(x_i, y_j, V_{i,j}, \Delta^-V_{i,j}\right) &= \sup_{rx_i + y_j \leqslant c_{i,j}^B \leqslant 1/\varepsilon}\left\{s_{i,j}^B\Delta^-V_{i,j} + \mathcal{F}\left(c_{i,j}^B, V_{i,j}\right)\right\}.\end{aligned} \tag{3.18}$$

This strategy of regularizing the Hamiltonian by truncations has been used in [16, 20].

Remark 3.2. *We observe that the FPK equations* (1.2) *(ii) are linear, hence their numerical approximation is standard and will not be studied in details here. We restrict ourselves to saying that, defining* (s^F, s^B) *as in* (3.14)*, the discrete scheme for* (1.2) *(ii) is in the form:*

$$\frac{s_{i,j}^{F,*}g_{i,j} - s_{i,j}^{B,*}g_{i,j}}{\Delta x} + \lambda_j g_{i,j} = \frac{s_{i-1,j}^{F,*}g_{i-1,j} - s_{i+1,j}^{B,*}g_{i+1,j}}{\Delta x} + \lambda_{\bar{j}}g_{i,\bar{j}}. \tag{3.19}$$

For $x_i > \underline{x}$*,* $g_{i,j}$ *approximates the density* $g_j(x_i)$*. If* m_j *exhibits a Dirac mass at* $\underline{x}$*, then its weight* μ_j *is approximated by* $g_{0,j}\Delta x$*.*

3.2 First elements in the analysis of the scheme

We first give some results on the scheme which hold for all $\theta > 0$. Section 4 and Section 5 will respectively contain results particular to the cases $\theta \geqslant 1$ and $0 < \theta < 1$.

Lemma 3.3. *Assume* $\theta > 1$ *(resp.,* $0 < \theta < 1$*). The numerical Hamiltonian* $\boldsymbol{H}\left(x_i, y_j, V_{i,j}, \boldsymbol{p}^F, \boldsymbol{p}^B\right)$ *is decreasing (resp., increasing) in* $V_{i,j}$*: if* $U_{i,j} < V_{i,j} < 0$ *then* $\boldsymbol{H}\left(x_i, y_j, V_{i,j}, \boldsymbol{p}^F, \boldsymbol{p}^B\right) <$ *(resp.,* $>$*)* $\boldsymbol{H}\left(x_i, y_j, U_{i,j}, \boldsymbol{p}^F, \boldsymbol{p}^B\right)$.

Proof. We only give the proof in the case $\theta \geqslant 1$, the proof in the other case being very similar. Let $c_{i,j}^{F,*}$ and $c_{i,j}^{B,*}$ be the maximizers given by (3.7). If $\underline{x} < x_i < \bar{x}$, then $\mathcal{F}\left(c_{i,j}^{B,*}, V_{i,j}\right) < \mathcal{F}\left(c_{i,j}^{B,*}, U_{i,j}\right)$ follows from $U_{i,j} < V_{i,j}$ and (2.9). From (3.7) we know $c_{i,j}^{F,*} \leqslant \bar{c}_j(x_i)$, a straightforward computation with (2.3) leads to

$$\mathcal{F}\left(c_{i,j}^{F,*}, V_{i,j}\right) - \mathcal{F}\left(\bar{c}_j(x_i), V_{i,j}\right) \leqslant \mathcal{F}\left(c_{i,j}^{F,*}, U_{i,j}\right) - \mathcal{F}\left(\bar{c}_j(x_i), U_{i,j}\right).$$

This yields

$$
\begin{aligned}
&\boldsymbol{H}\left(x_i, y_j, V_{i,j}, \boldsymbol{p}^F, \boldsymbol{p}^B\right) \\
&< s_{i,j}^{F,*}\boldsymbol{p}^F + s_{i,j}^{B,*}\boldsymbol{p}^B + \mathcal{F}\left(c_{i,j}^{B,*}, U_{i,j}\right) + \mathcal{F}\left(c_{i,j}^{F,*}, U_{i,j}\right) - \mathcal{F}\left(\bar{c}_j(x_i), U_{i,j}\right) \\
&\leqslant \boldsymbol{H}\left(x_i, y_j, U_{i,j}, \boldsymbol{p}^F, \boldsymbol{p}^B\right).
\end{aligned} \tag{3.20}
$$

At the boundaries $x_i = \underline{x}$ or $x_i = \bar{x}$, the conclusion is obvious from (3.3) and (3.4). □

The same result holds for the regularized Hamiltonian defined in (3.15), the proof being essentially the same as that of Lemma 3.3.

Lemma 3.4. *Assume $\theta > 1$ (resp., $0 < \theta < 1$). The numerical Hamiltonian $\boldsymbol{H}_\varepsilon\left(x_i, y_j, V_{i,j}, \boldsymbol{p}^F, \boldsymbol{p}^B\right)$ is decreasing (resp., increasing) in $V_{i,j}$: if $U_{i,j} < V_{i,j} < 0$ then $\boldsymbol{H}_\varepsilon\left(x_i, y_j, V_{i,j}, \boldsymbol{p}^F, \boldsymbol{p}^B\right) <$ (resp., $>$) $\boldsymbol{H}_\varepsilon\left(x_i, y_j, U_{i,j}, \boldsymbol{p}^F, \boldsymbol{p}^B\right)$.*

We now introduce the discrete sub and supersolution to (3.11), analogous to the functions introduced in Definition 2.1.

Definition 3.5. *We say that $\mathsf{U} = (\mathsf{U}_1, \mathsf{U}_2)$ is a discrete subsolution of (3.11) if for all i, j such that $i = 0, \cdots, I$ and $j \in \{1, 2\}$,*

$$
\frac{\rho}{\theta}\mathsf{U}_{i,j} \leqslant \boldsymbol{H}\left(x_i, y_j, \mathsf{U}_{i,j}, \Delta^+\mathsf{U}_{i,j}, \Delta^-\mathsf{U}_{i,j}\right) + \lambda_j(\mathsf{U}_{i,\bar{j}} - \mathsf{U}_{i,j}). \tag{3.21}
$$

Respectively, $\mathsf{V} = (\mathsf{V}_1, \mathsf{V}_2)$ is a discrete supersolution of (3.11) if for all i, j such that $i = 0, \cdots, I$ and $j \in \{1, 2\}$,

$$
\frac{\rho}{\theta}\mathsf{V}_{i,j} \geqslant \boldsymbol{H}\left(x_i, y_j, \mathsf{V}_{i,j}, \Delta^+\mathsf{V}_{i,j}, \Delta^-\mathsf{V}_{i,j}\right) + \lambda_j(\mathsf{V}_{i,\bar{j}} - \mathsf{V}_{i,j}). \tag{3.22}
$$

In what follows, we introduce discrete barriers as in Proposition 2.2. They will provide bounds on the numerical solutions.

Proposition 3.6. *With the constant b defined in Table 1, we consider the grid functions*

$$
\check{U}_{i,1} = \check{U}_{i,2} = \frac{(rx_i + y_1)^{1-\gamma}}{1-\gamma}, \quad \check{V}_{i,1} = \check{V}_{i,2} = \frac{(b(x_i + y_2/r))^{1-\gamma}}{1-\gamma}.
$$

The pairs $(\check{U}_{i,1}, \check{U}_{i,2})$ and $(\check{V}_{i,1}, \check{V}_{i,2})$ are respectively a sub- and a supersolution of (3.11).

Proof. Part 1: subsolution. A direct computation shows $\mathcal{F}\left(\bar{c}_1(x_i), \check{U}_{i,1}\right) = \frac{\rho}{\theta}\check{U}_{i,1}$. We first establish inequality (3.21) at $j = 1$. Let us set $c_{i,1}^F = c_{i,1}^B = \bar{c}_1(x_i)$, hence $s_{i,1}^F = s_{i,1}^B = 0$ if $0 < i < I$. Similarly, we set $s_{0,1}^F = s_{I,1}^B = 0$. Then,

$$
\begin{aligned}
&\boldsymbol{H}\left(x_i, y_1, \check{U}_{i,1}, \Delta^+\check{U}_{i,1}, \Delta^-\check{U}_{i,1}\right) \\
&\geqslant s_{i,1}^F\Delta^+\check{U}_{i,1} + s_{i,1}^B\Delta^-\check{U}_{i,1} + \mathcal{F}\left(c_{i,1}^F, \check{U}_{i,1}\right) + \mathcal{F}\left(c_{i,1}^B, \check{U}_{i,1}\right) - \mathcal{F}\left(\bar{c}_{i,1}, \check{U}_{i,1}\right) \\
&= \mathcal{F}\left(\bar{c}_1(x_i), \check{U}_{i,1}\right) = \frac{\rho}{\theta}\check{U}_{i,1}.
\end{aligned}
$$

Let us now turn to inequality (3.21) at $j = 2$. Set $c_{i,2}^F = c_{i,2}^B = \bar{c}_2(x_i) = rx_i + y_2$, hence $s_{i,2}^F = s_{i,2}^B = 0$, then

$$
\boldsymbol{H}\left(x_i, y_2, \check{U}_{i,2}, \Delta^+\check{U}_{i,2}, \Delta^-\check{U}_{i,2}\right) \geqslant \mathcal{F}\left(\bar{c}_2(x_i), \check{U}_{i,2}\right) \geqslant \mathcal{F}\left(\bar{c}_1(x_i), \check{U}_{i,2}\right) \geqslant \frac{\rho}{\theta}\check{U}_{i,2}.
$$

Therefore, for both $j = 1$ and $j = 2$, $\lambda_j(\check{U}_{i,\bar{j}} - \check{U}_{i,j}) = 0$, and

$$\frac{\rho}{\theta}\check{U}_{i,j} \leqslant \boldsymbol{H}\left(x_i, y_j, \check{U}_{i,j}, \Delta^+\check{U}_{i,j}, \Delta^-\check{U}_{i,j}\right) + \lambda_j(\check{U}_{i,\bar{j}} - \check{U}_{i,j}).$$

Part 2: supersolution. We aim at proving the inequality (3.22) at $j = 2$. From $\check{V}_{i,j} = \check{\mathsf{v}}_j(x_i)$, where $\check{\mathsf{v}}_j$ is defined in (2.11), we know $\bar{p}_j\left(x_i, \check{\mathsf{v}}_2(x_i)\right) = \bar{p}_j\left(x_i, \check{V}_{i,2}\right)$. Straightforward computation leads to $\frac{\rho}{\theta}\check{\mathsf{v}}_2(x_i) = H(x_i, y_2, \check{\mathsf{v}}_2(x_i), D\check{\mathsf{v}}_2(x_i))$, for $x_i > \underline{x}$. Since $\check{\mathsf{v}}_2$ is strictly concave, we can infer that

$$\Delta^+\check{V}_{i,2} < D\check{\mathsf{v}}_2(x_i) < \Delta^-\check{V}_{i,2}. \tag{3.23}$$

From Lemma 3.1 and (3.23), it follows

$$\begin{aligned} H(x_i, y_2, \check{\mathsf{v}}_2(x_i), D\check{\mathsf{v}}_2(x_i)) &= \boldsymbol{H}\left(x_i, y_2, \check{V}_{i,2}, D\check{\mathsf{v}}_2(x_i), D\check{\mathsf{v}}_2(x_i)\right) \\ &\geqslant \boldsymbol{H}\left(x_i, y_2, \check{V}_{i,2}, \Delta^+\check{V}_{i,2}, \Delta^-\check{V}_{i,2}\right), \end{aligned}$$

and we obtain

$$\frac{\rho}{\theta}\check{V}_{i,2} \geqslant \boldsymbol{H}\left(x_i, y_2, \check{V}_{i,2}, \Delta^+\check{V}_{i,2}, \Delta^-\check{V}_{i,2}\right).$$

Finally we prove the inequality (3.22) at $j = 1$. From $y_2 > y_1$ and $\check{V}_{i,1} = \check{V}_{i,2}$, we immediately obtain that

$$\boldsymbol{H}\left(x_i, y_2, \check{V}_{i,2}, \Delta^+\check{V}_{i,2}, \Delta^-\check{V}_{i,2}\right) > \boldsymbol{H}\left(x_i, y_1, \check{V}_{i,1}, \Delta^+\check{V}_{i,1}, \Delta^-\check{V}_{i,1}\right),$$

hence $\frac{\rho}{\theta}\check{V}_{i,1} \geqslant \boldsymbol{H}\left(x_i, y_1, \check{V}_{i,1}, \Delta^+\check{V}_{i,1}, \Delta^-\check{V}_{i,1}\right)$. ☐

Since for any $U_{i,j}$, $\boldsymbol{H}_\varepsilon\left(x_i, y_j, U_{i,j}, \Delta^+U_{i,j}, \Delta^-U_{i,j}\right) \leqslant \boldsymbol{H}\left(x_i, y_j, U_{i,j}, \Delta^+U_{i,j}, \Delta^-U_{i,j}\right)$, we have the following results.

Lemma 3.7. *A subsolution of* (3.16) *is also a subsolution of* (3.11). *A supersolution of* (3.11) *is a supersolution of* (3.16).

Proposition 3.8. *The grid functions $\check{U}$ and $\check{V}$, defined in Proposition 3.6, are respectively a constrained sub- and supersolution of* (3.16).

Proof. The fact that $\check{V}$ is supersolution to (3.16) follows directly from Lemma 3.7. To show that $\check{U}$ is subsolution to (3.16), we only need to observe that $c^F_{i,1} = c^B_{i,1} = \bar{c}_1(x_i)$ is still an admissible control, then we follow the same proof as that of Proposition 3.6. ☐

4 The case $\theta \geqslant 1$: existence, iterative algorithm and error estimate

Throughout this section we make the standing assumption $\theta \geqslant 1$.

4.1 Existence, uniqueness and gradient estimates

The comparison principle that follows is the discrete counterpart to Proposition 2.3.

Proposition 4.1. *If U and V are respectively a sub and supersolution of* (3.11), *then $U \leq V$.*

Proof. Suppose $U_{i*,j*} - V_{i*,j*} = \max_{i,j}\{U_{i,j} - V_{i,j}\} = \delta > 0$. Since U is a subsolution of (3.11),

$$\frac{\rho}{\theta} U_{i*,j*} \leqslant \boldsymbol{H}\left(x_i, y_j, U_{i*,j*}, \Delta^+ V_{i*,j*}, \Delta^- V_{i*,j*}\right) + \lambda_{j*}(U_{i*,\bar{j}*} - U_{i*,j*}). \tag{4.1}$$

Observe similarly as in Proposition 4.1 that $\Delta^+ V_{i*,j*} \geqslant \Delta^+ U_{i*,j*}$ if $0 \leqslant i^* \leqslant I-1$, and $\Delta^- V_{i*,j*} \leqslant \Delta^- U_{i*,j*}$ if $1 \leqslant i^* \leqslant I$. Lemma 3.1 and (3.2) then yield

$$\begin{aligned} &\boldsymbol{H}\left(x_{i*}, y_{j*}, U_{i*,j*}, \Delta^+ U_{i*,j*}, \Delta^- U_{i*,j*}\right) \\ \leqslant\ &\boldsymbol{H}\left(x_{i*}, y_{j*}, U_{i*,j*}, \Delta^+ V_{i*,j*}, \Delta^- V_{i*,j*}\right). \end{aligned}$$

Since $\theta \geqslant 1$ and $U_{i*,j*} > V_{i*,j*}$, Lemma 3.3 leads to

$$\begin{aligned} &\boldsymbol{H}\left(x_{i*}, y_{j*}, U_{i*,j*}, \Delta^+ V_{i*,j*}, \Delta^- V_{i*,j*}\right) \\ \leqslant\ &\boldsymbol{H}\left(x_{i*}, y_{j*}, V_{i*,j*}, \Delta^+ V_{i*,j*}, \Delta^- V_{i*,j*}\right). \end{aligned} \tag{4.2}$$

This implies, together with $U_{i*,j*} - V_{i*,j*} \geqslant U_{i*,\bar{j}*} - V_{i*,\bar{j}*}$, that

$$\frac{\rho}{\theta} U_{i*,j*} \leqslant \boldsymbol{H}\left(x_{i*}, y_{j*}, V_{i*,j*}, \Delta^+ V_{i*,j*}, \Delta^- V_{i*,j*}\right) + \lambda_{j*}(V_{i*,\bar{j}*} - V_{i*,j*}). \tag{4.3}$$

With the supersolution inequality for V, we get $\rho\delta/\theta \leqslant 0$, a contradiction. □

The following barrier property is then deduced from Proposition 4.1 and Proposition 3.6.

Proposition 4.2. *Let (V_1, V_2) be a solution of (3.11). Then*

$$\check{U} \preceq V \preceq \check{V}. \tag{4.4}$$

We now introduce a discrete Perron's method, which gives the existence of a solution V to (3.11).

Proposition 4.3. *Suppose that for all i, j,*

$$U_{i,j} := \sup\left\{Z_{i,j} : (Z_1, Z_2) \text{ is a subsolution of (3.11) nondecreasing with respect to } i\right\} \tag{4.5}$$

Then $U = (U_1, U_2)$ is a solution of (3.11).

Proof. We first observe that U_j is nondecreasing with respect to i.

Let us first prove U is a subsolution. Given the grid node x_i and a positive number ε, (4.5) implies that there exists a subsolution (Z_1, Z_2) such that $Z_{i,j} > U_{i,j} - \varepsilon\Delta x$. Moreover, from the maximality of U we infer $U_{i+1,j} \geqslant Z_{i+1,j}$, $U_{i-1,j} \geqslant Z_{i-1,j}$, $U_{i,\bar{j}} \geqslant Z_{i,\bar{j}}$. This implies $\Delta^+ U_{i,j} \geqslant \Delta^+ Z_{i,j} - \varepsilon$ and $\Delta^- U_{i,j} \leqslant \Delta^- Z_{i,j} + \varepsilon$. From Lemma 3.1, we deduce

$$\boldsymbol{H}\left(x_i, y_j, U_{i,j}, \Delta^+ U_{i,j}, \Delta^- U_{i,j}\right) \geqslant \boldsymbol{H}\left(x_i, y_j, Z_{i,j} + \varepsilon\Delta x, \Delta^+ Z_{i,j} - \varepsilon, \Delta^- Z_{i,j} + \varepsilon\right).$$

From the local Lipschitz continuity of $\boldsymbol{H}$ and letting $\varepsilon \to 0$, we deduce that U_j satisfies the subsolution inequality at x_i.

Next suppose U is not a solution to (3.11), i.e. there exists i^*, j^* and $\delta > 0$ such that

$$\begin{cases} \dfrac{\rho}{\theta} U_{i*,j*} - \boldsymbol{H}\left(x_{i*}, y_{j*}, U_{i*,j*}, \Delta^+ U_{i*,j*}, \Delta^- U_{i*,j*}\right) - \lambda_{j*}\left(U_{i*,\bar{j}*} - U_{i*,j*}\right) \leqslant -\delta, \\ \dfrac{\rho}{\theta} U_{i*,\bar{j}*} - \boldsymbol{H}\left(x_{i*}, y_{\bar{j}*}, U_{i*,\bar{j}*}, \Delta^+ U_{i*,\bar{j}*}, \Delta^- U_{i*,\bar{j}*}\right) - \lambda_{\bar{j}*}\left(U_{i*,j*} - U_{i*,\bar{j}*}\right) \leqslant 0. \end{cases} \tag{4.6}$$

We then make out two cases.

Case 1. $\Delta^+ U_{i*,j*} > 0$. It is possible to choose a constant $\eta > 0$ and a grid function $W_{i,j*}$ such that

$$W_{i,j} = U_{i,j} \ \forall i \neq i^*, \ W_{i*,j*} = U_{i*,j*} + \eta \Delta x \text{ and } W_{i*,j*} \leqslant U_{i*+1,j*}. \tag{4.7}$$

Plugging this information into the second line of (4.6) gives

$$\begin{aligned} &\frac{\rho}{\theta} U_{i,\bar{j}*} - \boldsymbol{H}\left(x_{i*}, y_{\bar{j}*}, U_{i*,\bar{j}*}, \Delta^+ U_{i*,\bar{j}*}, \Delta^- U_{i*,\bar{j}*}\right) - \lambda_{\bar{j}*}\left(W_{i*,j*} - U_{i*,\bar{j}*}\right) \\ \leqslant & -\lambda_{\bar{j}*} \eta \Delta x < 0. \end{aligned} \tag{4.8}$$

It is always possible to choose η sufficiently small such that $\left(\frac{\rho}{\theta} + \lambda_{j*}\right) \eta \Delta x \leqslant \delta/4$, hence

$$\frac{\rho}{\theta} W_{i*,j*} - \lambda_{j*}\left(U_{i*,\bar{j}*} - W_{i*,j*}\right) \leqslant \frac{\rho}{\theta} U_{i*,j*} - \lambda_{j*}\left(U_{i*,\bar{j}*} - U_{i*,j*}\right) + \frac{\delta}{4}. \tag{4.9}$$

Since $\Delta^+ W_{i*,j*} = \Delta^+ U_{i*,j*} - \eta$, $\Delta^- W_{i*,j*} = \Delta^- U_{i*,j*} + \eta$, we obtain for sufficiently small η that

$$\begin{aligned} & -\boldsymbol{H}\left(x_{i*}, y_{j*}, W_{i*,j*}, \Delta^+ W_{i*,j*}, \Delta^- W_{i*,j*}\right) \\ \leqslant & -\boldsymbol{H}\left(x_{i*}, y_{j*}, U_{i*,j*}, \Delta^+ U_{i*,j*}, \Delta^- U_{i*,j*}\right) + \frac{\delta}{4}. \end{aligned} \tag{4.10}$$

Summing up (4.9) and (4.10), we obtain from the first equation in (4.6) that

$$\frac{\rho}{\theta} W_{i*,j*} - \boldsymbol{H}\left(x_{i*}, y_{j*}, W_{i*,j*}, \Delta^+ W_{i*,j*}, \Delta^- W_{i*,j*}\right) - \lambda_{j*}\left(U_{i*,\bar{j}*} - W_{i*,j*}\right) \leqslant -\frac{\delta}{2}. \tag{4.11}$$

From (4.8) and (4.11), $(W_j, U_{\bar{j}*})$ is a subsolution while $W_{i*,j*} > U_{i*,j*}$. This contradicts the definition of U, see (4.5).

Case 2. $\Delta^+ U_{i*,j*} = 0$. We need to make out several sub-cases.

Case 2.1. Suppose $\Delta^+ U_{i*,j*} = 0$, $i^* = I - 1$. Then, $H^{\downarrow}\left(x_I, y_{j*}, U_{I,j*}, \Delta^- U_{I,j*}\right) = 0$. We derive from (4.6) that

$$\frac{\rho}{\theta} U_{I,j*} - \lambda_{j*}\left(U_{I,\bar{j}*} - U_{I,j*}\right) \leqslant -\delta$$

Defining W_{j*} as in (4.7), we obtain $\Delta^- W_{I,j*} = \eta$. We can choose η sufficiently small such that (4.9) holds at $i^* = I$ and $-H^{\downarrow}\left(x_I, y_{j*}, W_{I,j*}, \eta\right) \leqslant \frac{\delta}{4}$. This yields

$$\frac{\rho}{\theta} W_{I,j*} - \lambda_{j*}\left(U_{i*,\bar{j}*} - W_{i*,j*}\right) - H^{\downarrow}\left(x_I, y_{j*}, W_{I,j*}, \eta\right) \leqslant -\frac{\delta}{2},$$

i.e. the same contradiction as in *Case 1.*

Case 2.2. Suppose $\Delta^+ U_{i*,j*} = 0$, $i^* < I-1$ and there exists $l \leqslant I-1-i^*$, such that $\Delta^+ U_{i*+l,j*} > 0$ and

$$U_{i*,j*} = \cdots = U_{i*+l,j*}. \tag{4.12}$$

Since $U_{\bar{j}*}$ is nondecreasing w.r.t. i, $U_{i*+l,\bar{j}*} \geqslant U_{i*,\bar{j}*}$. With (4.12), we deduce

$$-\lambda_{j*}\left(U_{i*+l,\bar{j}*} - U_{i*+l,j*}\right) \leqslant -\lambda_{j*}\left(U_{i*,\bar{j}*} - U_{i*,j*}\right). \tag{4.13}$$

Since $\Delta^+ U_{i^*+l,j^*} > \Delta^+ U_{i^*,j^*}$ and $\Delta^- U_{i^*+l,j^*} = 0 \leqslant \Delta^- U_{i^*,j^*}$. Lemma 3.1 then yields

$$\begin{aligned} &\boldsymbol{H}\left(x_{i^*+l}, y_{j^*}, U_{i^*+l,j^*}, \Delta^+ U_{i^*+l,j^*}, \Delta^- U_{i^*+l,j^*}\right) \\ \geqslant\ &\boldsymbol{H}\left(x_{i^*}, y_{j^*}, U_{i^*+l,j^*}, \Delta^+ U_{i^*+l,j^*}, \Delta^- U_{i^*+l,j^*}\right) \\ \geqslant\ &\boldsymbol{H}\left(x_{i^*}, y_{j^*}, U_{i^*,j^*}, \Delta^+ U_{i^*,j^*}, \Delta^- U_{i^*,j^*}\right). \end{aligned} \tag{4.14}$$

Combining (4.12), (4.13) and (4.14), we obtain

$$\begin{aligned} &\frac{\rho}{\theta} U_{i^*+l,j^*} - \boldsymbol{H}\left(x_{i^*+l}, y_{j^*}, U_{i^*+l,j^*}, \Delta^+ U_{i^*+l,j^*}, \Delta^- U_{i^*+l,j^*}\right) \\ &- \lambda_{j^*}\left(U_{i^*+l,\bar{j}^*} - U_{i^*+l,j^*}\right) \leqslant -\delta. \end{aligned}$$

We can then consider the perturbation

$$W_{i,j} = U_{i,j}\ \forall i \neq i^* + l,\ W_{i^*+l,j^*} = U_{i^*+l,j^*} + \eta \Delta x \text{ and } W_{i^*+l,j^*} \leqslant U_{i^*+l+1,j^*}.$$

Similarly to *Case 1*, we obtain $(W_{j^*}, U_{\bar{j}^*})$ is a subsolution while $W_{i^*+l,j^*} > U_{i^*+l,j^*}$.
Case 2.3. Suppose $\Delta^+ U_{i^*,j^*} = 0$, $i^* < I - 1$ and the l defined in *Case 2.2.* does not exist, i.e. $U_{i^*,j^*} = \cdots = U_{I,j^*}$. Then $\Delta^+ U_{I-1,j^*} = 0$, $(U_{j^*}, U_{\bar{j}^*})$ satisfies (4.6) at I and the desired result follows from *Case 2.1.* □

Next, we prove that the grid functions $V_{i,j}$ are strictly increasing w.r.t. i.

Proposition 4.4. *Let V be a solution to* (3.11)*. The finite difference $\Delta^- V_{i,j}$ is strictly positive for all $i \geqslant 1$ and $j \in \{1, 2\}$.*

Proof. From the coercivity of the discrete Hamiltonian, $\Delta^- V_{i,j} \geqslant 0$. We argue by contradiction to show $\Delta^- V_{i,j} \neq 0$ for all (i, j). Suppose there exists (i^*, j^*), $i^* \geqslant 1$, such that $\Delta^- V_{i^*,j^*} = 0$, then from

$$H^{\uparrow}\left(x_{i^*}, y_{j^*}, V_{i^*,j^*}, \Delta^+ V_{i^*,j^*}\right) - H_{\min}\left(x_{i^*}, y_{j^*}, V_{i^*,j^*}\right) \geqslant 0$$

and $H^{\downarrow}\left(x_{i^*}, y_{j^*}, V_{i^*,j^*}, 0\right) = 0$ we deduce that

$$\boldsymbol{H}\left(x_{i^*}, y_{j^*}, V_{i^*,j^*}, \Delta^+ V_{i^*,j^*}, \Delta^- V_{i^*,j^*}\right) \geqslant 0. \tag{4.15}$$

By multiplying the equation (3.11) at i^*, j^* and $i^*, \bar{j}^*$ respectively by $\lambda_{\bar{j}^*}$ and λ_{j^*}, summing up, we obtain

$$\begin{aligned} \frac{\rho}{\theta}\left(\lambda_{\bar{j}^*} V_{i^*,j^*} + \lambda_{j^*} V_{i^*,\bar{j}^*}\right) &= \lambda_{j^*} \boldsymbol{H}\left(x_{i^*}, y_{\bar{j}^*}, V_{i^*,\bar{j}^*}, \Delta^+ V_{i^*,\bar{j}^*}, \Delta^- V_{i^*,\bar{j}^*}\right) \\ &\quad + \lambda_{\bar{j}^*} \boldsymbol{H}\left(x_{i^*}, y_{j^*}, V_{i^*,j^*}, \Delta^+ V_{i^*,j^*}, \Delta^- V_{i^*,j^*}\right). \end{aligned}$$

Then, we obtain from (4.15) and Proposition 4.2 that

$$\begin{aligned} \lambda_{j^*} \boldsymbol{H}\left(x_{i^*}, y_{\bar{j}^*}, V_{i^*,\bar{j}^*}, \Delta^+ V_{i^*,\bar{j}^*}, \Delta^- V_{i^*,\bar{j}^*}\right) &\leqslant \frac{\rho}{\theta}\left(\lambda_{\bar{j}^*} V_{i^*,j^*} + \lambda_{j^*} V_{i^*,\bar{j}^*}\right) \\ &\leqslant \frac{\rho}{\theta}\left(\lambda_1 + \lambda_2\right) \frac{(b(x_{i^*} + y_2/r))^{1-\gamma}}{1-\gamma} < 0, \end{aligned}$$

hence $\Delta^- V_{i^*,\bar{j}^*} > 0$. Subtracting the following two equations,

$$\begin{aligned}\frac{\rho}{\theta} V_{i^*,j^*} &= \boldsymbol{H}\left(x_{i^*}, y_{j^*}, V_{i^*,j^*}, \Delta^+ V_{i^*,j^*}, \Delta^- V_{i^*,j^*}\right) + \lambda_{j^*}\left(V_{i^*,\bar{j}^*} - V_{i^*,j^*}\right),\\ \frac{\rho}{\theta} V_{i^*-1,j^*} &= \boldsymbol{H}\left(x_{i^*-1}, y_{j^*}, V_{i^*-1,j^*}, \Delta^+ V_{i^*-1,j^*}, \Delta^- V_{i^*-1,j^*}\right)\\ &\quad + \lambda_{j^*}\left(V_{i^*-1,\bar{j}^*} - V_{i^*-1,j^*}\right),\end{aligned}$$

and taking into account that $\Delta^- V_{i^*,j^*} = 0$, we obtain

$$\begin{aligned}-\lambda_{j^*}\Delta^- V_{i^*,\bar{j}^*} = \boldsymbol{H}&\left(x_{i^*}, y_{j^*}, V_{i^*,j^*}, \Delta^+ V_{i^*,j^*}, \Delta^- V_{i^*,j^*}\right)\\ &- \boldsymbol{H}\left(x_{i^*-1}, y_{j^*}, V_{i^*-1,j^*}, \Delta^+ V_{i^*-1,j^*}, \Delta^- V_{i^*-1,j^*}\right).\end{aligned} \tag{4.16}$$

For brevity, let us name RHS the right hand side of (4.16). We are going to see $RHS \geqslant 0$. If $i^* > 1$, then

$$\begin{aligned}&RHS\\ &= H^{\downarrow}\left(x_{i^*}, y_{j^*}, V_{i^*,j^*}, \Delta^- V_{i^*,j^*}\right) - H^{\downarrow}\left(x_{i^*-1}, y_{j^*}, V_{i^*-1,j^*}, \Delta^- V_{i^*-1,j^*}\right) && (I.1)\\ &\quad + H^{\uparrow}\left(x_{i^*}, y_{j^*}, V_{i^*,j^*}, \Delta^+ V_{i^*,j^*}\right) - H_{\min}\left(x_{i^*}, y_{j^*}, V_{i^*,j^*}\right) && (I.2)\\ &\quad - H^{\uparrow}\left(x_{i^*-1}, y_{j^*}, V_{i^*-1,j^*}, \Delta^+ V_{i^*-1,j^*}\right) + H_{\min}\left(x_{i^*-1}, y_{j^*}, V_{i^*-1,j^*}\right) && (I.3)\end{aligned}$$

From $\Delta^- V_{i^*,j^*} = 0 \leqslant \Delta^- V_{i^*-1,j^*}$ and (3.5), we deduce

$$H^{\downarrow}\left(x_{i^*}, y_{j^*}, V_{i^*,j^*}, \Delta^- V_{i^*,j^*}\right) = 0, \quad H^{\downarrow}\left(x_{i^*-1}, y_{j^*}, V_{i^*-1,j^*}, \Delta^- V_{i^*-1,j^*}\right) \leqslant 0,$$

hence $(I.1) \geqslant 0$. It is obvious $(I.2) \geqslant 0$, and we deduce from $\Delta^+ V_{i^*-1,j^*} = \Delta^- V_{i^*,j^*} = 0$ that $(I.3) = 0$. This yields $RHS \geqslant 0$.

If $i^* = 1$, then $\Delta^- V_{1,j^*} = \Delta^+ V_{0,j^*} = (V_{1,j^*} - V_{0,j^*})/\Delta x = 0$, and

$$\begin{aligned}RHS &= H^{\uparrow}\left(x_1, y_{j^*}, V_{1,j^*}, \Delta^+ V_{1,j^*}\right) - H^{\uparrow}\left(\underline{x}, y_{j^*}, V_{0,j^*}, 0\right)\\ &= H^{\uparrow}\left(x_1, y_{j^*}, V_{0,j^*}, \Delta^+ V_{1,j^*}\right) - H_{\min}\left(\underline{x}, y_{j^*}, V_{0,j^*}\right) \geqslant 0.\end{aligned}$$

If $i^* = I$, then $\Delta^- V_{I,j^*} = (V_{I,j^*} - V_{I-1,j^*})/\Delta x = 0$ and

$$RHS = H^{\downarrow}\left(x_I, y_{j^*}, V_{I,j^*}, 0\right) - H^{\downarrow}\left(x_{I-1}, y_{j^*}, V_{I-1,j^*}, \Delta^- V_{I-1,j^*}\right) \geqslant 0.$$

We conclude by observing $RHS \geqslant 0$, contradicts the fact that $\Delta^- V_{i^*,\bar{j}^*} > 0$. □

Corollary 4.5. *For any Δx and for any solution V to* (3.11)*, there exists $\varsigma > 0$ such that $\Delta^- V_{i,j} \geqslant \varsigma$ for all $i \geqslant 1$ and $j \in \{1, 2\}$.*

Proposition 4.6. *Let V be a solution to* (3.11)*. There exists a constant $C > 0$ uniform in Δx such that, for all (i, j), $i \geqslant 1$ and $j \in \{1, 2\}$, $\Delta^+ V_{i-1,j} = \Delta^- V_{i,j} \leqslant C$.*

Proof. From Proposition 4.2, we know that the grid function V is bounded from below by $\check{U}_{0,1}$ and from above by $\check{V}_{I,2}$. Hence, there exists a constant $\tilde{C} > 0$, uniform with respect to Δx, such that any solution V to (3.11) satisfies $p_{\min}(x_i, y_j, V_{i,j}) \leqslant \tilde{C}$. Therefore, it is not restrictive to assume that the constant in the statement of Proposition 4.6 is larger than $\tilde{C}$. This allows us to focus on the pairs of indices (i, j), $0 \leqslant i \leqslant I-1$ and $j \in \{1, 2\}$, such that $\Delta^+ V_{i,j} \geqslant \tilde{C} > p_{\min}(x_i, y_j, V_{i,j})$, thus

$$H\left(x_i, y_j, V_{i,j}, \Delta^+ V_{i,j}\right) = H^{\uparrow}\left(x_i, y_j, V_{i,j}, \Delta^+ V_{i,j}\right) \leqslant \frac{\rho}{\theta} V_{i,j} - \lambda_j\left(V_{i,\bar{j}} - V_{i,j}\right).$$

We conclude by using the above mentioned bound on V and the coercivity of the Hamiltonian. □

4.2 A Howard-Newton algorithm

With the ς bound in Corollary 4.5, we consider an iterative method for solving the regularized HJB equation (3.16) instead of (3.11) since they have the same unique solution if $1/\varepsilon > \rho^{\psi}\varsigma^{-\psi}\left((1-\gamma)\check{V}_{I,2}\right)^{\frac{1-\gamma\psi}{1-\gamma}}$ ($\check{V}$ is the supersolution defined in Proposition 3.6). The Howard-Newton iterative algorithm is described in Algorithm 1.

Algorithm 4.1 Howard-Newton algorithm

Initialize $c_{i,j}^{F,(0)} = rx_i + y_j,\ c_{i,j}^{B,(0)} = rx_i + y_j$, and set $n = 0$

repeat

  Update the saving policy: $s_{i,j}^{F,(n)} = rx_i + y_j - c_{i,j}^{F,(n)}, \quad s_{i,j}^{B,(n)} = rx_i + y_j - c_{i,j}^{B,(n)}$.

  Update the value function by solving with Newton's method:

$$\begin{aligned}\frac{\rho}{\theta}V_{i,j}^{(n)} = {} & s_{i,j}^{F,(n)}\Delta^{+}V_{i,j}^{(n)} + s_{i,j}^{B,(n)}\Delta^{-}V_{i,j}^{(n)} + \mathcal{F}\left(c_{i,j}^{F,(n)}, V_{i,j}^{(n)}\right) + \mathcal{F}\left(c_{i,j}^{B,(n)}, V_{i,j}^{(n)}\right) \\ & - \mathcal{F}\left(\bar{c}_j(x_i), V_{i,j}^{(n)}\right) + \lambda_j\left(V_{i,\bar{j}}^{(n)} - V_{i,j}^{(n)}\right).\end{aligned} \tag{4.17}$$

  Update the consumption policy:

$$\begin{cases} c_{i,j}^{F,(n+1)} = \min\left\{\rho^{\psi}\left(\Delta^{+}V_{i,j}^{(n)}\right)_{+}^{-\psi}\left((1-\gamma)V_{i,j}^{(n)}\right)^{\frac{1-\gamma\psi}{1-\gamma}}, \bar{c}_j(x_i)\right\} \text{ for } x_i < \bar{x}, \\ c_{I,j}^{F,(n+1)} = r\bar{x} + y_j, \\ c_{i,j}^{B,(n+1)} = \max\left\{\min\left\{1/\varepsilon, \rho^{\psi}\left(\Delta^{-}V_{i,j}^{(n)}\right)_{+}^{-\psi}\left((1-\gamma)V_{i,j}^{(n)}\right)^{\frac{1-\gamma\psi}{1-\gamma}}\right\}, \bar{c}_j(x_i)\right\} \\ \text{for } x_i > \underline{x}, \quad c_{0,j}^{B,(n+1)} = r\underline{x} + y_j. \end{cases} \tag{4.18}$$

  $n \leftarrow n+1$

until $\sum_j\left(\max_i\left|c_{i,j}^{F,(n+1)} - c_{i,j}^{F,(n)}\right| + \max_i\left|c_{i,j}^{B,(n+1)} - c_{i,j}^{B,(n)}\right|\right) < \mathrm{Tol}_c$.

Lemma 4.7. *Consider the admissible set of controls* $\mathcal{C}_{i,j}^{F,(n+1)} = [0, \bar{c}_j(x_i)]$ *and* $\mathcal{C}_{i,j}^{B,(n+1)} = [\bar{c}_j(x_i), 1/\varepsilon]$. *Then for* $n \geqslant 0$,

$$\begin{aligned} c_{i,j}^{F,(n+1)} &= \underset{c\in\mathcal{C}_{i,j}^{F,(n+1)}}{\arg\max}\left\{-c\Delta^{+}V_{i,j}^{(n)} + \mathcal{F}\left(c, V_{i,j}^{(n)}\right)\right\}, \\ c_{i,j}^{B,(n+1)} &= \underset{c\in\mathcal{C}_{i,j}^{B,(n+1)}}{\arg\max}\left\{-c\Delta^{-}V_{i,j}^{(n)} + \mathcal{F}\left(c, V_{i,j}^{(n)}\right)\right\}. \end{aligned}$$

We will see in the proof of Theorem 4.8 that the set for $\mathcal{C}_{i,j}^{B,(n+1)}$ is indeed nonempty for all n.

Theorem 4.8. *The sequence* $V^{(n)}$ *generated by Algorithm 1 converges to the unique solution of* (3.11).

Proof. Step 1. We show that $V^{(n)}$ is bounded above uniformly w.r.t. n. From (4.17), we know $(V_{i,1}^{(n)}, V_{i,2}^{(n)})$ is a subsolution of (3.16). We infer from Proposition 3.8 and Proposition 4.1 that $V_{i,j}^{(n)} \leqslant \check{V}_{i,j}$ for all n. This ensures that $\left((1-\gamma)V_{i,j}^{(n)}\right)^{\frac{1-\gamma\psi}{1-\gamma}}$ is always well defined and bounded.

Step 2. We claim that $V^{(0)} \preceq V^{(1)}$. Indeed, suppose by contradiction that

$$V^{(0)}_{i*,j*} - V^{(1)}_{i*,j*} = \max_{i,j}\{V^{(0)}_{i,j} - V^{(1)}_{i,j}\} = \delta > 0. \tag{4.19}$$

This implies

$$\lambda_{j*}\left(V^{(0)}_{i*,\bar{j}*} - V^{(0)}_{i*,j*}\right) \leqslant \lambda_{j*}\left(V^{(1)}_{i*,\bar{j}*} - V^{(1)}_{i*,j*}\right), \tag{4.20}$$

$$\Delta^+ V^{(1)}_{i*,j*} \geqslant \Delta^+ V^{(0)}_{i*,j*} \text{ if } i^* < I, \quad \Delta^- V^{(1)}_{i*,j*} \leqslant \Delta^- V^{(0)}_{i*,j*} \text{ if } i^* > 0. \tag{4.21}$$

We consider three cases.

Case 1: $0 < i^* < I$. From the initialization of Algorithm 1, we know

$$\frac{\rho}{\theta} V^{(0)}_{i,j} = \mathcal{F}\left(\bar{c}_j(x_i), V^{(0)}_{i,j}\right) + \lambda_j\left(V^{(0)}_{i,\bar{j}} - V^{(0)}_{i,j}\right). \tag{4.22}$$

It is easy to verify that $(\check{U}_{0,1}, \check{U}_{0,2})$ is a subsolution to (4.22), hence $V^{(0)}_{i,j} \geqslant \check{U}_{0,j}$. We deduce from (4.17), (4.18) and Lemma 4.7 that

$$\begin{aligned} \frac{\rho}{\theta} V^{(0)}_{i*,j*} \leqslant{}& s^{F,(1)}_{i*,j*}\Delta^+ V^{(0)}_{i*,j*} + s^{B,(1)}_{i*,j*}\Delta^- V^{(0)}_{i*,j*} + \mathcal{F}\left(c^{F,(1)}_{i*,j*}, V^{(0)}_{i*,j*}\right) + \mathcal{F}\left(c^{B,(1)}_{i*,j*}, V^{(0)}_{i*,j*}\right) \\ &- \mathcal{F}\left(\bar{c}_j(x_{i*}), V^{(0)}_{i*,j*}\right) + \lambda_{j*}\left(V^{(0)}_{i*,\bar{j}*} - V^{(0)}_{i*,j*}\right). \end{aligned} \tag{4.23}$$

From (4.19) and same argument as in Lemma 3.3,

$$\begin{aligned} &\mathcal{F}\left(c^{F,(1)}_{i*,j*}, V^{(0)}_{i*,j*}\right) + \mathcal{F}\left(c^{B,(1)}_{i*,j*}, V^{(0)}_{i*,j*}\right) - \mathcal{F}\left(\bar{c}_j(x_{i*}), V^{(0)}_{i*,j*}\right) \\ \leqslant{}& \mathcal{F}\left(c^{F,(1)}_{i*,j*}, V^{(1)}_{i*,j*}\right) + \mathcal{F}\left(c^{B,(1)}_{i*,j*}, V^{(1)}_{i*,j*}\right) - \mathcal{F}\left(\bar{c}_j(x_{i*}), V^{(1)}_{i*,j*}\right). \end{aligned} \tag{4.24}$$

By construction, $s^{F,(1)}_{i*,j*} \geqslant 0$ and $s^{B,(1)}_{i*,j*} \leqslant 0$. Then (4.21) implies that

$$s^{F,(1)}_{i*,j*}\Delta^+ V^{(0)}_{i*,j*} + s^{B,(1)}_{i*,j*}\Delta^- V^{(0)}_{i*,j*} \leqslant s^{F,(1)}_{i*,j*}\Delta^+ V^{(1)}_{i*,j*} + s^{B,(1)}_{i*,j*}\Delta^- V^{(1)}_{i*,j*}. \tag{4.25}$$

From (4.20), (4.24), (4.25) and the equation (4.17) for $V^{(1)}$, we get $\rho\delta/\theta \leqslant 0$, a contradiction.

Case 2: $i^* = 0$. The same argument holds, provided (4.23) is replaced with

$$\frac{\rho}{\theta} V^{(0)}_{0,j*} \leqslant s^{F,(1)}_{0,j*}\Delta^+ V^{(0)}_{0,j*} + \mathcal{F}\left(c^{F,(1)}_{0,j*}, V^{(0)}_{0,j*}\right) + \lambda_{j*}\left(V^{(1)}_{0,\bar{j}*} - V^{(1)}_{0,j*}\right).$$

The conclusion follows similarly as in the previous case.

Case 3: $i^* = I$. The same argument holds, provided (4.23) is replaced with

$$\frac{\rho}{\theta} V^{(0)}_{I,j*} \leqslant s^{B,(1)}_{I,j*}\Delta^- V^{(0)}_{I,j*} + \mathcal{F}\left(c^{B,(1)}_{I,j*}, V^{(0)}_{I,j*}\right) + \lambda_{j*}\left(V^{(1)}_{I,\bar{j}*} - V^{(1)}_{I,j*}\right),$$

and the desired result follows.

Step 3. Following the proof of *Step 2*, we can obtain by induction $V^{(n)} \preceq V^{(n+1)}$.

Therefore, $V^{(n)}$ is an increasing sequence w.r.t. n and is bounded above uniformly in n, yielding the desired result. □

4.3 Convergence rate to the unique viscosity solution when $\theta \geqslant 1$

We denote by (v_1, v_2) the unique solution of (2.2), and (V_1, V_2) the unique solution of the discrete problem (3.11) on $\mathcal{G}^{\Delta x}$.

Lemma 4.9. *We make the same assumptions as in Proposition 2.5, then there exists a constant C (uniform in Δx) such that*

$$\max\left(\max_{x\in\mathcal{G}^{\Delta x},\, x>\underline{x}} \left\{\Delta^- v_2(x) - Dv_2(x)\right\},\ \max_{x\in\mathcal{G}^{\Delta x},\, x<\bar{x}} \left\{Dv_2(x) - \Delta^+ v_2(x)\right\}\right) \leqslant C\Delta x. \tag{4.26}$$

There exists η sufficiently small (uniformly in Δx) such that

$$\max_{x\in\mathcal{G}^{\Delta x},\, \underline{x}<x<\eta} \left\{\Delta^- v_1(x) - Dv_1(x)\right\} \leqslant C\sqrt{\Delta x}, \tag{4.27}$$

$$\max_{x\in\mathcal{G}^{\Delta x},\, \underline{x}<x<\eta} \left\{-s_1(x)\left(\Delta^- v_1(x) - Dv_1(x)\right)\right\} \leqslant C\Delta x. \tag{4.28}$$

Proof. In the proof, the constant C may change from line to line but is always independent of Δx. From Proposition 2.4, we know that $v_j \in W^{2,\infty}_{loc}(\underline{x}, \bar{x}]$. Since $s_2(\underline{x}) > 0$, $D^2 v_2(\underline{x})$ is bounded. This implies $D^2 v_2 \in L^\infty(\underline{x}, \bar{x})$ and there exists C_2 (independent of Δx) such that

$$\max\left(\max_{x\in\mathcal{G}^{\Delta x},\, x>\underline{x}} \left\{\Delta^- v_2(x) - Dv_2(x)\right\},\ \max_{x\in\mathcal{G}^{\Delta x},\, x<\bar{x}} \left\{Dv_2(x) - \Delta^+ v_2(x)\right\}\right) \leqslant C_2\Delta x.$$

To prove (4.27) and (4.28), we consider two cases: $x = \underline{x} + \Delta x$ and $\underline{x} + \Delta x < x < \eta$.

Case 1: $x = \underline{x} + \Delta x$. From the strict concavity of v_1,

$$0 < \Delta^- v_1(x + \Delta x) - Dv_1(x + \Delta x) < Dv_1(x) - Dv_1(x + \Delta x) \leqslant C\sqrt{\Delta x},$$

where for the last inequality we used (2.14). From (2.17), we know $0 < -s_1(x + \Delta x) \leqslant C\sqrt{\Delta x}$, hence $-s_1(x + \Delta x)\left(\Delta^- v_1(x + \Delta x) - Dv_1(x + \Delta x)\right) \leqslant C\Delta x$.

Case 2: $\underline{x} + 2\Delta x \leqslant x < \eta$. By using a Taylor expansion with integral remainder, we deduce

$$\Delta^- v_1(x) - Dv_1(x) = -\frac{1}{\Delta x}\int_{x-\Delta x}^{x} (z - (x - \Delta x)) D^2 v_1(z) dz.$$

From $x - \Delta x \leqslant z < \eta$ and Lemma 2.10, we deduce that $0 < -D^2 v_1(z) < \frac{C}{\sqrt{z-\underline{x}}} \leqslant \frac{C}{\sqrt{x-\Delta x-\underline{x}}} \leqslant \frac{C}{\sqrt{\Delta x}}$, where for the last inequality we used $x \geqslant \underline{x} + 2\Delta x$. Hence,

$$\Delta^- v_1(x) - Dv_1(x) \leqslant \frac{C}{(\Delta x)^{1/2}}\frac{1}{\Delta x}\int_{x-\Delta x}^{x} (z - (x - \Delta x)) dz \leqslant C(\Delta x)^{1/2}.$$

Since $0 < -s_1(x) \leqslant C\sqrt{x - \underline{x}}$ for $\underline{x} + 2\Delta x \leqslant x < \eta$,

$$\begin{aligned} -s_1(x)\left(\Delta^- v_1(x) - Dv_1(x)\right) &= \frac{-s_1(x)}{\Delta x}\int_{x-\Delta x}^{x} (z - (x - \Delta x))\left(-D^2 v_1(z)\right) dz \\ &\leqslant \frac{C\sqrt{x - \underline{x}}}{\sqrt{x - \Delta x - \underline{x}}}\frac{1}{\Delta x}\int_{x-\Delta x}^{x} (z - (x - \Delta x)) dz \\ &\leqslant \frac{C(\sqrt{x - \Delta x - \underline{x}} + \sqrt{\Delta x})}{\sqrt{x - \Delta x - \underline{x}}}\Delta x \leqslant C\Delta x. \end{aligned}$$

For the last inequality, we observe that $\frac{\sqrt{\Delta x}}{\sqrt{x-\Delta x-\underline{x}}} \leqslant 1$ since $x \geqslant \underline{x} + 2\Delta x$.

□

Theorem 4.10. *We make the same assumptions as in Proposition 2.5, and $\bar{x} \in \mathcal{G}^{\Delta x}$ is sufficiently large such that $s_2(\bar{x}) < 0$. Then for sufficiently small Δx, there exists a constant $C > 0$, independent of Δx, such that*

$$\max_{j,\, x \in \mathcal{G}^{\Delta x}} |v_j(x) - V_j(x)| \leqslant C\Delta x. \tag{4.29}$$

Proof. In this proof, C denotes a generic positive constant that may change from line to line.

Step 1. Let us define

$$\sigma = \max_{j,\, x \in \mathcal{G}^{\Delta x}} \{v_j(x) - V_j(x)\} = v_{j^*}(x^*) - V_{j^*}(x^*), \tag{4.30}$$

and suppose $\sigma > 0$. Since $v_j \in C^1(\underline{x}, +\infty)$, we know that if $x^* > \underline{x}$ then

$$\frac{\rho}{\theta} v_{j^*}(x^*) = H\left(x^*, y_{j^*}, v_{j^*}(x^*), Dv_{j^*}(x^*)\right) + \lambda_{j^*}(v_{\bar{j}^*}(x^*) - v_{j^*}(x^*)). \tag{4.31}$$

Since Dv_j is uniformly continuous in $[\underline{x}, +\infty)$, (4.31) holds if $x^* = \underline{x}$.

Since $V_{j^*}(x^* + \Delta x) - V_{j^*}(x^*) \geqslant v_{j^*}(x^* + \Delta x) - v_{j^*}(x^*)$ if $\underline{x} \leqslant x^* < \bar{x}$, $\Delta^+ V_{j^*}(x^*) \geqslant \Delta^+ v_{j^*}(x^*)$ if $\underline{x} \leqslant x^* < \bar{x}$. Similarly, $\Delta^- V_{j^*}(x^*) \leqslant \Delta^- v_{j^*}(x^*)$ if $\underline{x} < x^* \leqslant \bar{x}$. From Lemma 3.1,

$$\begin{aligned} &\boldsymbol{H}\left(x^*, y_{j^*}, V_{j^*}(x^*), \Delta^+ V_{j^*}(x^*), \Delta^- V_{j^*}(x^*)\right) \\ \geqslant\ &\boldsymbol{H}\left(x^*, y_{j^*}, V_{j^*}(x^*), \Delta^+ v_{j^*}(x^*), \Delta^- v_{j^*}(x^*)\right) \\ \geqslant\ &\boldsymbol{H}\left(x^*, y_{j^*}, v_{j^*}(x^*), \Delta^+ v_{j^*}(x^*), \Delta^- v_{j^*}(x^*)\right). \end{aligned} \tag{4.32}$$

For the last inequality, we used $\sigma > 0$ and Lemma 3.3. Observe that if $x^* = \underline{x}$ or $x^* = \bar{x}$, (4.32) holds with $\boldsymbol{H}$ given by (3.2). We infer from (3.11) and (4.32) that

$$\frac{\rho}{\theta} V_{j^*}(x^*) \geqslant \boldsymbol{H}\left(x^*, y_{j^*}, v_{j^*}(x^*), \Delta^+ v_{j^*}(x^*), \Delta^- v_{j^*}(x^*)\right) + \lambda_{j^*}(V_{\bar{j}^*}(x^*) - V_{j^*}(x^*)). \tag{4.33}$$

From (4.30), $v_{j^*}(x^*) - V_{j^*}(x^*) \geqslant v_{\bar{j}^*}(x^*) - V_{\bar{j}^*}(x^*)$. Therefore, by subtracting (4.31) from (4.33) we obtain

$$\begin{aligned} &\frac{\rho}{\theta}\left(v_{j^*}(x^*) - V_{j^*}(x^*)\right) \\ \leqslant\ &H\left(x^*, y_{j^*}, v_{j^*}(x^*), Dv_{j^*}(x^*)\right) - \boldsymbol{H}\left(x^*, y_{j^*}, v_{j^*}(x^*), \Delta^+ v_{j^*}(x^*), \Delta^- v_{j^*}(x^*)\right). \end{aligned} \tag{4.34}$$

If $j^* = 2$, then from (4.26) and the consistency of the numerical Hamiltonian $\boldsymbol{H}$, it follows $\frac{\rho}{\theta}\left(v_2(x^*) - V_2(x^*)\right) \leqslant C\Delta x$.

We now estimate the right hand side of (4.34), with $j^* = 1$, by making out three cases.

Case 1. $\underline{x} < x^* < \eta$, where η has been introduced in Lemma 4.9. Since $s_1(x^*) < 0$, we know $Dv_1(x^*) < p_{\min}(x^*, y_1, v_1(x^*))$, and $\Delta^- v_1(x^*) < p_{\min}(x^*, y_1, v_1(x^*))$ follows from the continuity of Dv_1, and

$$\begin{aligned} &H\left(x^*, y_1, v_1(x^*), Dv_1(x^*)\right) - \boldsymbol{H}\left(x^*, y_1, v_1(x^*), \Delta^+ v_1(x^*), \Delta^- v_1(x^*)\right) \\ =\ &H\left(x^*, y_1, v_1(x^*), Dv_1(x^*)\right) - H\left(x^*, y_1, v_1(x^*), \Delta^- v_1(x^*)\right) \\ \leqslant\ &(rx^* + y_1)\left(Dv_1(x^*) - \Delta^- v_1(x^*)\right) \\ &+ \underbrace{\frac{\rho^\psi \left(Dv_1(x^*)\right)^{1-\psi}}{\psi - 1}\left((1-\gamma)v_1(x^*)\right)^{\frac{1-\gamma\psi}{1-\gamma}} - \frac{\rho^\psi \left(\Delta^- v_1(x^*)\right)^{1-\psi}}{\psi - 1}\left((1-\gamma)v_1(x^*)\right)^{\frac{1-\gamma\psi}{1-\gamma}}}_{(I)} \end{aligned}$$

From the mean value theorem, there exists $\xi \in (0,1)$ such that

$$
\begin{aligned}
& (I) \\
= & \left(-\rho^{\psi}\left(\xi Dv_1(x^*) + (1-\xi)\Delta^- v_1(x^*)\right)^{-\psi}\right)\left((1-\gamma)v_1(x^*)\right)^{\frac{1-\gamma\psi}{1-\gamma}} \cdot \\
& \left(Dv_1(x^*) - \Delta^- v_1(x^*)\right) \\
= & -\rho^{\psi}\left(Dv_1(x^*)\right)^{-\psi}\left((1-\gamma)v_1(x^*)\right)^{\frac{1-\gamma\psi}{1-\gamma}}\left(Dv_1(x^*) - \Delta^- v_1(x^*)\right) \\
& + \rho^{\psi}\underbrace{\left(\left(\xi Dv_1(x^*) + (1-\xi)\Delta^- v_1(x^*)\right)^{-\psi} - \left(Dv_1(x^*)\right)^{-\psi}\right)}_{(II)}\underbrace{\left((1-\gamma)v_1(x^*)\right)^{\frac{1-\gamma\psi}{1-\gamma}}}_{III} \cdot \\
& \left(\Delta^- v_1(x^*) - Dv_1(x^*)\right) \\
\leqslant & c_1(x^*)\left(\Delta^- v_1(x^*) - Dv_1(x^*)\right) \\
& + \rho^{\psi}\left(Dv_1(\bar{x})\right)^{-\psi-1}\left(b\left(\bar{x} + \frac{y_2}{r}\right)\right)^{1-\gamma\psi}\left(\Delta^- v_1(x^*) - Dv_1(x^*)\right)^2,
\end{aligned}
$$

where we used (2.8) for the last inequality. To bound (II) for the last inequality, we infer from the concavity of v_1 that $\Delta^- v_1(x^*) > Dv_1(x^*) > Dv_1(\bar{x}) > 0$, hence

$$(II) \leqslant \psi\left(Dv_1(\bar{x})\right)^{-\psi-1}\left(\Delta^- v_1(x^*) - Dv_1(x^*)\right).$$

For (III), we have used Proposition 2.2 and the comparison principle.

Since $s_1(x^*) = rx^* + y_1 - c_1(x^*)$,

$$
\begin{aligned}
& H\left(x^*, y_1, v_1(x^*), Dv_1(x^*)\right) - \boldsymbol{H}\left(x^*, y_1, v_1(x^*), \Delta^+ v_1(x^*), \Delta^- v_1(x^*)\right) \\
\leqslant & -s_1(x^*)\left(\Delta^- v_1(x^*) - Dv_1(x^*)\right) + C\left(\Delta^- v_1(x^*) - Dv_1(x^*)\right)^2 \leqslant C\Delta x,
\end{aligned}
$$

where for the last inequality we used (4.27) and (4.28) in Lemma 4.9.

Case 2. $x^* = \underline{x}$. Since $s_1(\underline{x}) = 0$, $Dv_1(\underline{x}) = p_{\min}\left(\underline{x}, y_1, v_1(\underline{x})\right)$ and $\Delta^+ v_1(\underline{x}) < p_{\min}\left(\underline{x}, y_1, v_1(\underline{x})\right)$, hence $H\left(\underline{x}, y_1, v_1(\underline{x}), Dv_1(\underline{x})\right) - H^{\uparrow}\left(\underline{x}, y_1, V_1(\underline{x}), \Delta^+ v_1(\underline{x})\right) = 0$.

Case 3. $x^* \in [\eta, \bar{x}]$. Since $s_1(x^*) < 0$,

$$
\begin{aligned}
& H\left(x^*, y_1, v_1(x^*), Dv_1(x^*)\right) - \boldsymbol{H}\left(x^*, y_1, v_1(x^*), \Delta^+ v_1(x^*), \Delta^- v_1(x^*)\right) \\
= & H\left(x^*, y_1, v_1(x^*), Dv_1(x^*)\right) - H\left(x^*, y_1, v_1(x^*), \Delta^- v_1(x^*)\right) \leqslant C\Delta x
\end{aligned}
$$

For the last inequality, we used $v_1 \in W^{2,\infty}(\eta, \bar{x})$.

Since we argue with sufficiently small Δx, we may assume $\Delta x < 1$ without loss of generality. For all the three cases above, we can use (4.34) and finally obtain that $\sigma \leqslant \frac{C\theta}{\rho}\Delta x$.

Step 2. Now we reverse the direction of the estimate and define

$$\sigma_2 = \sup_{j,\ x \in \mathcal{G}^{\Delta x}} \{V_j(x) - v_j(x)\}.$$

The proof that $\sigma_2 \leqslant C\Delta x$ is similar and we omit the details. □

5 Solution method in the case $0 < \theta < 1$

Throughout this section we make the standing assumption $0 < \theta < 1$. Consider the map $(V_1, V_2) = \Gamma_\varepsilon(\widetilde{V}_1, \widetilde{V}_2)$ defined by

$$\begin{cases} \dfrac{\rho}{\theta} V_{i,1} = \boldsymbol{H}_\varepsilon \left(x_i, y_1, \widetilde{V}_{i,1}, \Delta^+ V_{i,1}, \Delta^- V_{i,1} \right) + \lambda_1 \left(V_{i,2} - V_{i,1} \right), \\ \dfrac{\rho}{\theta} V_{i,2} = \boldsymbol{H}_\varepsilon \left(x_i, y_2, \widetilde{V}_{i,2}, \Delta^+ V_{i,2}, \Delta^- V_{i,2} \right) + \lambda_2 \left(V_{i,1} - V_{i,2} \right). \end{cases} \tag{5.1}$$

Any fixed point of Γ_ε is a solution to the regularized HJB equation (3.16). The following comparison principle is the discrete form of Proposition 2.11.

Proposition 5.1. *Assume $\check{U} \preceq \widetilde{V} \preceq \check{V}$. Let U and V be respectively a sub- and supersolution of (5.1), then $U \preceq V$.*

Proof. We argue by contradiction. Suppose $U_{i*,j*} - V_{i*,j*} = \max_{i,j}\{U_{i,j} - V_{i,j}\} = \delta > 0$. We denote $\bar{\jmath}^* = 3 - j^*$ and observe $U_{i*,j*} - V_{i*,j*} \geqslant U_{i*,\bar{\jmath}*} - V_{i*,\bar{\jmath}*}$, hence

$$\lambda_{j*}(U_{i*,\bar{\jmath}*} - U_{i*,j*}) \leqslant \lambda_{j*}(V_{i*,\bar{\jmath}*} - V_{i*,j*}). \tag{5.2}$$

Since $U_{i*+1,j*} - V_{i*+1,j*} \leqslant U_{i*,j*} - V_{i*,j*}$, $\quad U_{i*-1,j*} - V_{i*-1,j} \leqslant U_{i*,j*} - V_{i*,j*}$, we infer that $\Delta^+ V_{i*,j*} \geqslant \Delta^+ U_{i*,j*}$, $\Delta^- V_{i*,j*} \leqslant \Delta^- U_{i*,j*}$. From Lemma 3.1 we deduce

$$\begin{aligned} &\boldsymbol{H}_\varepsilon \left(x_{i*}, y_{j*}, \widetilde{V}_{i*,j*}, \Delta^+ U_{i*,j*}, \Delta^- U_{i*,j*} \right) \\ \leqslant\ &\boldsymbol{H}_\varepsilon \left(x_{i*}, y_{j*}, \widetilde{V}_{i*,j*}, \Delta^+ V_{i*,j*}, \Delta^- V_{i*,j*} \right). \end{aligned} \tag{5.3}$$

and

$$\frac{\rho}{\theta} U_{i*,j*} \leqslant \boldsymbol{H}_\varepsilon \left(x_i, y_j, \widetilde{V}_{i*,j*}, \Delta^+ V_{i*,j*}, \Delta^- V_{i*,j*} \right) + \lambda_{j*}(U_{i*,\bar{\jmath}*} - U_{i*,j*}). \tag{5.4}$$

Since V is a supersolution,

$$\frac{\rho}{\theta} V_{i*,j*} \geqslant \boldsymbol{H}_\varepsilon \left(x_i, y_j, \widetilde{V}_{i*,j*}, \Delta^+ V_{i*,j*}, \Delta^- V_{i*,j*} \right) + \lambda_{j*}(V_{i*,\bar{\jmath}*} - V_{i*,j*}). \tag{5.5}$$

Subtracting (5.5) from (5.4) and using (5.2), we get $\rho\delta/\theta \leqslant 0$, a contradiction. ☐

We observe that, since the value $\widetilde{V}_{i,j}$ is fixed in (5.1), Proposition 5.1 holds for all $\theta > 0$. A major difficulty in the case $\theta < 1$ is that Proposition 4.1 may not hold. Therefore, we cannot directly use the comparison principle in Proposition 4.1 and obtain $\check{U} \preceq V \preceq \check{V}$, for all solution V to (3.11), as in the case $\theta \geqslant 1$. However, the following results will allow us to look for a solution V such that $\check{U} \preceq V \preceq \check{V}$.

Lemma 5.2. *We have the barrier properties*

(i). If $(U_1, U_2) = \Gamma_\varepsilon(\check{U}_1, \check{U}_2)$, then $U \succeq \check{U}$.

(ii). If $(V_1, V_2) = \Gamma_\varepsilon(\check{V}_1, \check{V}_2)$, then $V \preceq \check{V}$.

Proof. (i). If $\widetilde{V}_{i,j} = \check{U}_{i,j}$, $(\check{U}_1, \check{U}_2)$ itself is a subsolution to the HJB equation (5.1). From Proposition 5.1, $U_{i,j} \geqslant \check{U}_{i,j}$.

(ii). Similarly, if $\widetilde{V}_{i,j} = \check{V}_{i,j}$ then $(\check{V}_1, \check{V}_2)$ itself is a supersolution to the HJB equation (5.1). From Proposition 5.1, $V_{i,j} \leqslant \check{V}_{i,j}$. ☐

Proposition 5.3. *The map Γ_ε is monotone: if $(U_1, U_2) = \Gamma_\varepsilon(\widetilde{U}_1, \widetilde{U}_2)$, $(V_1, V_2) = \Gamma_\varepsilon(\widetilde{V}_1, \widetilde{V}_2)$ and $\check{U} \preceq \widetilde{U} \preceq \widetilde{V} \preceq \check{V}$, then $U \preceq V$.*

Proof. From $\check{U} \preceq \widetilde{U} \preceq \widetilde{V} \preceq \check{V}$ and Lemma 3.4, we know for all (i, j),

$$\boldsymbol{H}_\varepsilon\left(x_i, y_j, \widetilde{U}_{i,j}, \Delta^+ U_{i,j}, \Delta^- U_{i,j}\right) \leqslant \boldsymbol{H}_\varepsilon\left(x_i, y_j, \widetilde{V}_{i,j}, \Delta^+ U_{i,j}, \Delta^- U_{i,j}\right),$$

hence (U_1, U_2) is a subsolution to the discrete HJB system $(V_1, V_2) = \Gamma_\varepsilon(\widetilde{V}_1, \widetilde{V}_2)$. Proposition 5.1 yields the desired result. □

We deduce the following invariance principle for Γ_ε, analogous to Proposition 2.14.

Proposition 5.4. *If $\check{U} \preceq \widetilde{V} \preceq \check{V}$ and $V = (V_1, V_2) = \Gamma_\varepsilon(\widetilde{V}_1, \widetilde{V}_2)$, then $\check{U} \preceq V \preceq \check{V}$.*

Proof. Let us denote $\widehat{V} = (\widehat{V}_1, \widehat{V}_2) = \Gamma_\varepsilon(\check{V}_1, \check{V}_2)$. From Proposition 5.3 and $\widetilde{V} \preceq \check{V}$, $V \preceq \widehat{V}$. From Lemma 5.2 we obtain $V \preceq \widehat{V} \preceq \check{V}$. The proof for $\check{U} \preceq V$ is similar. □

Definition 5.5. *The grid function V^ε is a solution to* (3.16) *if the equations are satisfied and $\check{U} \preceq V^\varepsilon \preceq \check{V}$. We say that $\underline{V}^\varepsilon$ is the minimal solution of* (3.16) *if it is a solution and $\underline{V}^\varepsilon \preceq V^\varepsilon$ for all V^ε solution to* (3.16).

Corollary 5.6. *If $\check{U} \preceq \widetilde{V} \preceq \underline{V}^\varepsilon$ and $(V_1, V_2) = \Gamma_\varepsilon(\widetilde{V}_1, \widetilde{V}_2)$ then $V \preceq \underline{V}^\varepsilon$.*

Taking advantage of the theoretical analysis above, we propose Algorithm 2 to solve the discrete HJB equation in the case $0 < \theta < 1$. Let $\bar{c}_j$ be defined as in (3.8).

Theorem 5.7. *The sequence $V^{(k)}$ generated by Algorithm 2 converges to the minimal solution $\underline{V}^\varepsilon$ of* (3.16).

Proof. Step 1. Solvability of the Howard inner loop. We first claim that for each $\left(V_{i,1}^{(k)}, V_{i,2}^{(k)}\right)$ such that $\check{U} \preceq V^{(k)} \preceq \check{V}$, the sequence $V^{(k,n)}$ converges to $V^{(k+1)}$. The proof is similar to that of Theorem 4.8, hence we skip it.

Step 2. Uniform boundedness of the sequence $V^{(k)}$. From Proposition 5.4, we deduce that if $\check{U} \preceq V^{(k)} \preceq \check{V}$ then $\check{U} \preceq V^{(k+1)} \preceq \check{V}$. By induction, $\check{U} \preceq V^{(k)} \preceq \check{V}$ holds for all k. This ensures that at each outer iteration the term $\left((1-\gamma)V_{i,j}^{(k)}\right)^{\frac{1-\gamma\psi}{1-\gamma}}$ is always well defined and bounded.

Step 3. Monotonicity and convergence. We first observe that Lemma 5.2 implies $V^{(1)} \succeq V^{(0)}$. From Proposition 5.3, if $V^{(k)} \succeq V^{(k-1)}$, then $V^{(k+1)} \succeq V^{(k)}$. Arguing by induction, this implies that $V^{(k)}$ is increasing. Since $V^{(k)} \preceq \check{V}$ holds for all k, $V^{(k)}$ converges to a limit $\widehat{V}$ such that $\widehat{V} \preceq \check{V}$.

Step 4. Minimality. From Definition 5.5, we know if there exists a solution V^ε, then $V^{(0)} = \check{U} \preceq V^\varepsilon$. From Corollary 5.6, an easy induction leads to $V^{(k)} \preceq V^\varepsilon$ for all k, hence $\widehat{V} \preceq V^\varepsilon$. From the design of the algorithm, $\widehat{V}$ is solution to (3.16), therefore $\widehat{V} = \underline{V}^\varepsilon$. □

Remark 5.8. *In Algorithm 1, we only need a strictly negative upper bound so that $\left((1-\gamma)V_{i,j}^{(n)}\right)^{\frac{1-\gamma\psi}{1-\gamma}}$ is well defined, since $\left((1-\gamma)V_{i,j}^{(n)}\right)^{\frac{1-\gamma\psi}{1-\gamma}} \to 0$ if $V_{i,j}^{(n)} \to -\infty$. In Algorithm 2, with $1 - \gamma\psi < 0$, we need both upper and lower bounds to ensure that $\left((1-\gamma)V_{i,j}^{(k)}\right)^{\frac{1-\gamma\psi}{1-\gamma}}$ is well defined and bounded.*

Algorithm 5.1 Howard-Tarski-Kantorovich algorithm: upward variant

Initialize $V_{i,j}^{(0)} = \check{U}_{i,j}$, and set $k = 0$ $\qquad \triangleright$ Outer loop

repeat

Use Howard algorithm to solve:

$$\begin{cases} \dfrac{\rho}{\theta} V_{i,1}^{(k+1)} = \boldsymbol{H}_\varepsilon \left(x_i, y_1, V_{i,1}^{(k)}, \Delta^+ V_{i,1}^{(k+1)}, \Delta^- V_{i,1}^{(k+1)} \right) + \lambda_1 \left(V_{i,2}^{(k+1)} - V_{i,1}^{(k+1)} \right), \\ \dfrac{\rho}{\theta} V_{i,2}^{(k+1)} = \boldsymbol{H}_\varepsilon \left(x_i, y_2, V_{i,2}^{(k)}, \Delta^+ V_{i,2}^{(k+1)}, \Delta^- V_{i,2}^{(k+1)} \right) + \lambda_2 \left(V_{i,1}^{(k+1)} - V_{i,2}^{(k+1)} \right). \end{cases} \tag{5.6}$$

repeat

Initialize $V_{i,j}^{(k,0)} = V_{i,j}^{(k)}$, and set $n = 0$ $\qquad \triangleright$ Inner loop

Update the policy

$$\begin{cases} c_{i,j}^{F,(k,n+1)} = \min \left\{ \rho^\psi \left(\Delta^+ V_{i,j}^{(k,n)} \right)_+^{-\psi} \left((1-\gamma) V_{i,j}^{(k)} \right)^{\frac{1-\gamma\psi}{1-\gamma}}, \bar{c}_j(x_i) \right\} \\ \text{for } x_i < \bar{x}, \quad c_{I,j}^{F,(n+1)} = r\bar{x} + y_j, \quad s_{i,j}^{F,(k,n+1)} = r x_i + y_j - c_{i,j}^{F,(k,n+1)}, \\ c_{i,j}^{B,(k,n+1)} = \max \left\{ \min \left\{ 1/\varepsilon, \rho^\psi \left(\Delta^- V_{i,j}^{(k,n)} \right)_+^{-\psi} \left((1-\gamma) V_{i,j}^{(k)} \right)^{\frac{1-\gamma\psi}{1-\gamma}} \right\}, \bar{c}_j(x_i) \right\} \\ \text{for } x_i > \underline{x}, \quad c_{0,j}^{B,(n+1)} = r\underline{x} + y_j, \quad s_{i,j}^{B,(k,n+1)} = r x_i + y_j - c_{i,j}^{B,(k,n+1)}. \end{cases} \tag{5.7}$$

Solve

$$\begin{aligned} \frac{\rho V_{i,j}^{(k,n+1)}}{\theta} &= s_{i,j}^{F,(k,n+1)} \Delta^+ V_{i,j}^{(k,n+1)} + s_{i,j}^{B,(k,n+1)} \Delta^- V_{i,j}^{(k,n+1)} \\ &\quad + \mathcal{F}\left(c_{i,j}^{F,(k,n+1)}, V_{i,j}^{(k)} \right) + \mathcal{F}\left(c_{i,j}^{B,(k,n+1)}, V_{i,j}^{(k)} \right) - \mathcal{F}\left(\bar{c}_j(x_i), V_{i,j}^{(k)} \right) \\ &\quad + \lambda_j \left(V_{i,\bar{j}}^{(k,n+1)} - V_{i,j}^{(k,n+1)} \right). \end{aligned} \tag{5.8}$$

$n \leftarrow n + 1$

until $\sum_j \left(\max_i \left| c_{i,j}^{F,(k,n+1)} - c_{i,j}^{F,(k,n)} \right| + \max_i \left| c_{i,j}^{B,(k,n+1)} - c_{i,j}^{B,(k,n)} \right| \right) < \mathrm{Tol}_c$

Set $V_{i,j}^{(k+1)} = V_{i,j}^{(k,n)}$.

$k \leftarrow k + 1$

until $\sum_j \max_i \frac{\left| V_{i,j}^{(k+1)} - V_{i,j}^{(k)} \right|}{1 + \left| V_{i,j}^{(k)} \right|} < \mathrm{Tol}_V$.

The next result deals with the monotone behavior of the solution to (5.1) as ε decreases.

Proposition 5.9. *Assume $\varepsilon > \varepsilon' > 0$ and $\check{U} \preceq \widetilde{V}^\varepsilon \preceq \widetilde{V}^{\varepsilon'} \preceq \check{V}$. Let V^ε and $V^{\varepsilon'}$ be respectively the solutions to* (5.1) *with $(V_1^\varepsilon, V_2^\varepsilon) = \Gamma_\varepsilon(\widetilde{V}_1^\varepsilon, \widetilde{V}_2^\varepsilon)$ and $(V_1^{\varepsilon'}, V_2^{\varepsilon'}) = \Gamma_{\varepsilon'}(\widetilde{V}_1^{\varepsilon'}, \widetilde{V}_2^{\varepsilon'})$. Then $V^\varepsilon \preceq V^{\varepsilon'}$.*

Proof. Since $\varepsilon > \varepsilon' > 0$ and $\check{U}_{i,j} \leqslant \widetilde{V}_{i,j}^\varepsilon \leqslant \widetilde{V}_{i,j}^{\varepsilon'} \leqslant \check{V}_{i,j}$,

$$\begin{aligned} \boldsymbol{H}_\varepsilon \left(x_i, y_j, \widetilde{V}_{i,j}^\varepsilon, \Delta^+ V_{i,j}^\varepsilon, \Delta^- V_{i,j}^\varepsilon \right) &\leqslant \boldsymbol{H}_{\varepsilon'} \left(x_i, y_j, \widetilde{V}_{i,j}^\varepsilon, \Delta^+ V_{i,j}^\varepsilon, \Delta^- V_{i,j}^\varepsilon \right) \\ &\leqslant \boldsymbol{H}_{\varepsilon'} \left(x_i, y_j, \widetilde{V}_{i,j}^{\varepsilon'}, \Delta^+ V_{i,j}^\varepsilon, \Delta^- V_{i,j}^\varepsilon \right), \end{aligned}$$

where we have used Lemma 3.4 for the last inequality. Therefore, V^ε is a subsolution of the equation satisfied by $V^{\varepsilon'}$, we apply Proposition 5.1 and obtain $V^\varepsilon \preceq V^{\varepsilon'}$. □

Let us now deal with the existence of solution $\underline{V}$ to (3.11), obtained as the limit of $\underline{V}^\varepsilon$ as $\varepsilon \to 0$.

Proposition 5.10. *If $\varepsilon > \varepsilon' > 0$, then $\underline{V}^\varepsilon \preceq \underline{V}^{\varepsilon'}$. Moreover, $\underline{V}^\varepsilon$ converges uniformly on $\mathcal{G}^{\Delta x}$ to the minimal solution of* (3.11) *as $\varepsilon \to 0$.*

Proof. Let us denote by $V^{(k),\varepsilon}$ the sequence of grid functions constructed by the Howard-Tarski-Kantorovich algorithm given the regularization parameter ε. Observe that the initial guess does not depend on ε, namely that $V^{(0),\varepsilon}_{i,j} = V^{(0),\varepsilon'}_{i,j} = \check{U}_{i,j}$. From Proposition 5.9, we infer that if $V^{(k),\varepsilon} \preceq V^{(k),\varepsilon'}$ then $V^{(k+1),\varepsilon} \preceq V^{(k+1),\varepsilon'}$. By induction, we obtain that $V^{(k),\varepsilon} \preceq V^{(k),\varepsilon'}$ for all $k \in \mathbb{N}$. Sending $k \to +\infty$ yields $\underline{V}^\varepsilon \preceq \underline{V}^{\varepsilon'}$. From Proposition 5.4, $\underline{V}^\varepsilon \preceq \check{V}$ for all ε. We thus conclude that the sequence $\underline{V}^\varepsilon$ converges uniformly on $\mathcal{G}^{\Delta x}$ to the minimal solution of (3.11). □

We observe that, having established the existence of a minimal solution, we can use the same proof as that of Proposition 4.4 to show $\Delta^- \underline{V}_{i,j} > \varsigma$ for all $i \geqslant 1$ when $0 < \theta < 1$. The regularized HJB equation (3.16) and (3.11) have the same minimal solution if $1/\varepsilon > \rho^\psi \varsigma^{-\psi} \left((1-\gamma)\check{U}_{0,1}\right)^{\frac{1-\gamma\psi}{1-\gamma}}$, where $\check{U}$ is defined in Proposition 3.6.

Proposition 5.11. *For $\varepsilon > 0$ sufficiently small, $\underline{V}^\varepsilon = \underline{V}$.*

Next, we introduce Algorithm 3, which is a variant of Algorithm 2 in which the outer loop starts with $(\check{V}_1, \check{V}_2)$. Then, $V^{(k)}$ is a non increasing sequence of grid functions.

Algorithm 5.2 Howard-Tarski-Kantorovich algorithm: downward variant

Initialize $V^{(0)}_{i,j} = \check{V}_{i,j}$, and set $k = 0$ ▷ Outer loop
repeat
  Use Howard algorithm to solve (5.6) for $V^{(k+1)}$
  repeat
    Initialize $c^{(k,0)}_{i,j} = rx_i + y_j$, and set $n = 0$ ▷ Inner loop
    Policy evaluation: solve (5.8) for $V^{(k,n+1)}$
    Update the policy with (5.7)
    $n \leftarrow n + 1$
  until $\sum_j \left(\max_i \left|c^{F,(k,n+1)}_{i,j} - c^{F,(k,n)}_{i,j}\right| + \max_i \left|c^{B,(k,n+1)}_{i,j} - c^{B,(k,n)}_{i,j}\right|\right) < \mathrm{Tol}_c$
  Set $V^{(k+1)}_{i,j} = V^{(k,n)}_{i,j}$.
  $k \leftarrow k + 1$
until $\sum_j \max_i \frac{\left|V^{(k+1)}_{i,j} - V^{(k)}_{i,j}\right|}{1+\left|V^{(k)}_{i,j}\right|} < \mathrm{Tol}_V$.

Theorem 5.12. *The sequence $V^{(k)}$ generated by Algorithm 3 converges to the maximal solution $\overline{V}^\varepsilon$ of* (3.16).

The proof is similar to that of Theorem 5.7, hence we omit the details. Observe that the design of Howard iteration inner loop is different. In Algorithm 2, $V^{(k,n)}$ is an increasing sequence for both indexes k and n. Therefore, we initialize $V_{i,j}^{(k,0)} = V_{i,j}^{(k)}$ at each inner loop of Algorithm 2 in order to accelerate convergence. In Algorithm 3, $V^{(k)}$ is decreasing while $V^{(k,n)}$ is increasing in n for each fixed k. Therefore, $V^{(k)}$ itself may no longer be a subsolution of (5.6) and we initialize the inner loop of Algorithm 3 using a feasible consumption policy.

Similarly to Proposition 5.10, we can obtain the maximal solution $\overline{V}$ as the limit of $\overline{V}^{\varepsilon}$ when $\varepsilon \to 0$. Due to the lack of comparison principle in the case $0 < \theta < 1$, we do not have uniqueness results for (2.2) or (3.11). Nevertheless, in all our experiments. we have observed that Algorithm 2 and Algorithm 3 converge to the same solution, i.e. $\underline{V} = \overline{V}$. This leads us to think that the solution to (3.11) is in fact unique also with $0 < \theta < 1$, even if this is not yet proved.

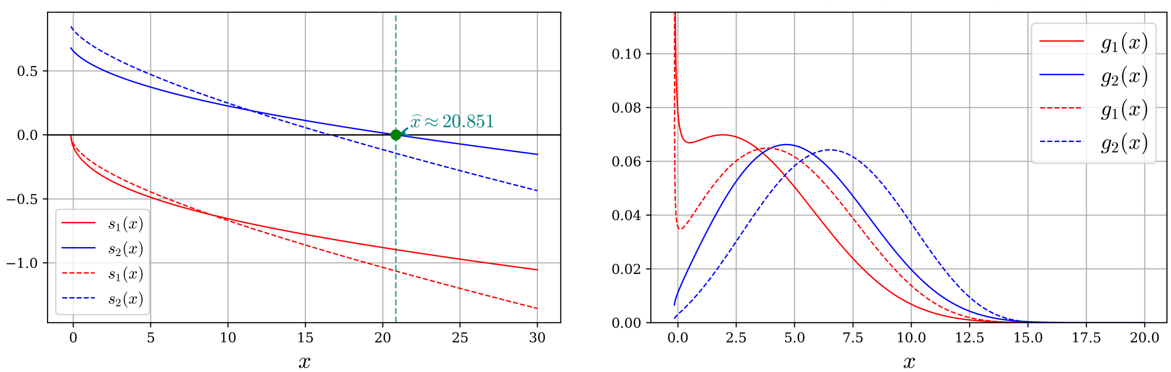
6 Numerical results

For a complete solution of the MFG system (1.2), including the computation of equilibrium interest rates, we refer the reader to [4]. In all numerical experiments, we apply uniform stopping criteria: $\text{Tol}_c = 10^{-7}$ and $\text{Tol}_V = 10^{-10}$. We fix the following parameters (as in [4]): $\rho = 0.05$, $y_1 = 0.5$, $y_2 = 1.5$, $\lambda_1 = 0.2$, $\lambda_2 = 0.2$, $\underline{x} = -0.15$ and consider the following tests. *Test 1 MFG*, with $\theta > 1$, is solved with Algorithm 1. *Test 2 MFG* and *Test 3 MFG*, with $0 < \theta < 1$, are both solved with Algorithm 2 and Algorithm 3.

Test 1 MFG: $\gamma = 2$, $\psi = 0.4$, $r^* = 0.0288$.

Test 2 MFG: $\gamma = 4$, $\psi = 0.5$, $r^* = 0.0266$. *Test 3 MFG:* $\gamma = 20$, $\psi = 0.5$, $r^* = 0.0086$.

Figure 1: Saving policy and asset distribution for Test 2 (solid) and 3 (dotted)



Next, we focus on the convergence analysis of the finite-difference scheme and the algorithmic performance of the HJB solver. To isolate these aspects, tests for which the interest rates are held fixed at the equilibrium values are referred to as *Test 1 HJB*, *Test 2 HJB* and *Test 3 HJB*.

For Test 1, Fig. 2 illustrates the fact that the sequence produced by Algorithm 1 is non decreasing as stated in Theorem 4.8, i.e. $V^{(n)} \preceq V^{(n+1)}$. For *Test 2 HJB*, Fig. 3 (resp. Fig. 4) illustrates the fact, stated in Theorem 5.7 (resp Theorem 5.12) , that Algorithm 2 (resp. Algorithm 3) produces a non decreasing (resp. non increasing) sequence of discrete functions.

We now consider an additional *Test 4 HJB* in which all parameters are identical to those in *Test 1 HJB*, with the exception that $\lambda_2 = 0.02$. Fig. 5 illustrates the results of Proposition 2.5.

Figure 2: Algorithm 1 constructs a non decreasing sequence of grid functions

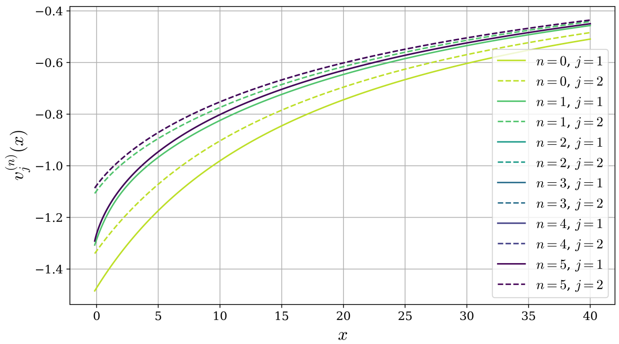


Figure 3: Algorithm 2 constructs a non decreasing sequence of grid functions

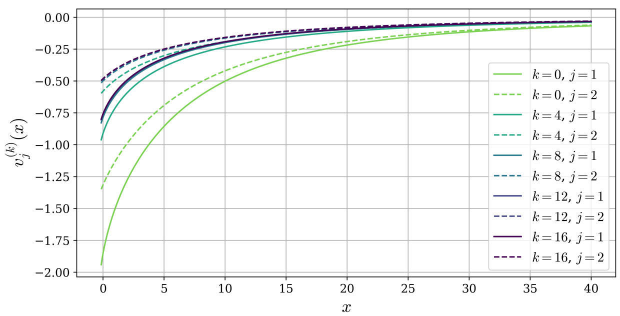
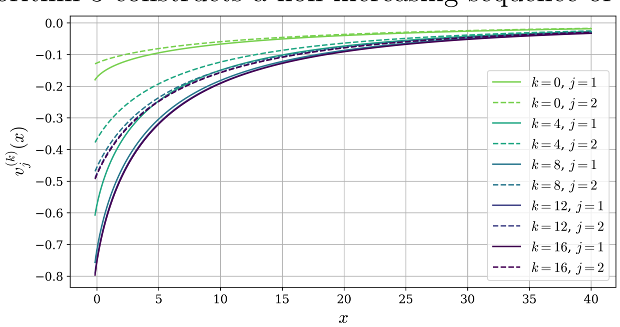


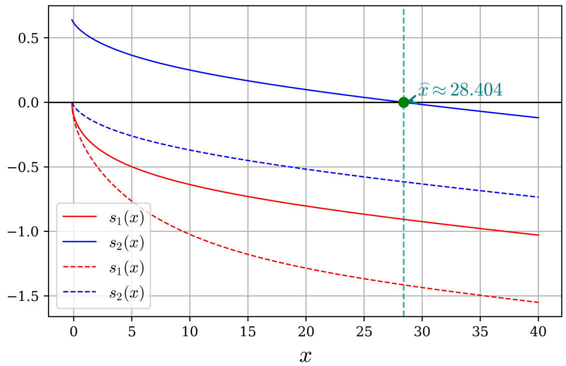
Figure 4: Algorithm 3 constructs a non increasing sequence of grid functions



In particular, we observe $s_2(\underline{x}) = 0$ in *Test 4 HJB*. Such a behavior would not occur if (2.12) is satisfied. In the case corresponding to *Test 4 HJB*, the agents associated with y_2 do not have precautionary motive to save if the transition risk λ_2 is too small. Note that our scheme still works for *Test 4 HJB*, although we have not analyzed this situation for brevity.

We next supplement the numerical results of *Test 1 HJB* with a numerical estimation of the convergence rate illustrating Section 4.3. Since the exact solution (v_1, v_2) of the HJB system is not available in closed form, we compute a reference solution $(V_1^{\text{ref}}, V_2^{\text{ref}})$ on a fine grid with $\Delta x_{\text{ref}} = 0.001$ and treat it as a proxy for the true solution. We then solve the scheme on a sequence of ten coarser grids, with Δx ranging from 0.002 to 0.1, spaced geometrically so that the grid spacings are uniformly distributed on a log scale. For each coarse grid spacing $\Delta x > \Delta x_{\text{ref}}$, the reference

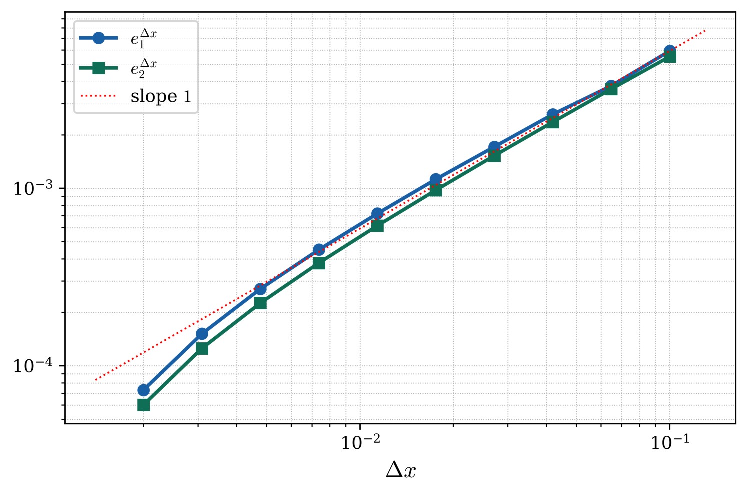
Figure 5: Saving policy in *Test 1 HJB* (solid) and *Test 4 HJB* (dotted)



solution is interpolated onto the coarse grid $\mathcal{G}^{\Delta x}$ via the piecewise linear interpolation $\mathbb{I}[\cdot]$. We use $e_j^{\Delta x} = \max_{j,\, x\in\mathcal{G}^{\Delta x}} \left| \mathbb{I}[V_j^{\text{ref}}](x) - V_j(x) \right|$ to substitute $\max_{j,\, x\in\mathcal{G}^{\Delta x}} |v_j(x) - V_j(x)|$.

Fig. 6 displays errors as functions of Δx on a log-log scale, together with a reference line of slope 1. Both error curves decrease monotonically as $\Delta x \to 0$, confirming the convergence of the scheme. Moreover, the curves run parallel to the slope-1 reference line, consistent with the theoretical bound established in Theorem 4.10.

Figure 6: convergence rate in *Test 1 HJB*



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